

MIXED METRIC DIMENSION OF CARTESIAN PRODUCT OF CYCLE AND PATH

(Dimensi Metrik Bercampur bagi Hasil Darab Cartesian antara Kitaran dan Laluan)

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ABSTRACT

The mixed metric dimension is a recently introduced parameter that measures how a set of vertices can uniquely identify all elements (both vertices and edges) of a graph. A vertex $w \in V$ distinguishes two elements $x, y \in V \cup E$ if their distances from w are distinct, i.e., $d_G(w, x) \neq d_G(w, y)$. A set $S \subseteq V$ is a mixed metric generator for a connected graph G if every pair of distinct elements in G is distinguished by at least one vertex in S . The minimum cardinality of such a set is the mixed metric dimension of G , denoted by $mdim(G)$. In essence, the selected vertices in S provide a unique representation for every vertex and edge in the graph. In this paper, we determine the mixed metric dimensions for the Cartesian product of a cycle and a path, denoted by $C_m \square P_n$.

Keywords: mixed metric dimension; mixed metric resolving set; Cartesian product of cycle and path

ABSTRAK

Dimensi metrik bercampur merupakan parameter yang baru diperkenalkan bagi mengukur bagaimana suatu set bucu dapat mengenal pasti secara unik semua unsur (kedua-dua bucu dan sisi) bagi sesebuah graf. Suatu bucu $w \in V$ membezakan dua unsur $x, y \in V \cup E$ jika jaraknya dari w adalah berbeza, iaitu $d_G(w, x) \neq d_G(w, y)$. Suatu set $S \subseteq V$ ialah penjana metrik bercampur bagi graf terhubung G jika setiap pasangan unsur yang berbeza dalam G dibezakan oleh sekurang-kurangnya satu bucu dalam S . Kardinaliti minimum bagi set tersebut ialah dimensi metrik bercampur bagi G , yang ditandakan sebagai $mdim(G)$. Pada hakikatnya, bucu-bucu yang dipilih dalam S menyediakan perwakilan unik bagi setiap bucu dan sisi dalam graf tersebut. Dalam makalah ini, kami menentukan dimensi metrik bercampur bagi hasil darab Cartesian antara kitaran dan laluan, yang ditandakan sebagai $C_m \square P_n$.

Kata kunci: dimensi metrik bercampur; set penyelesaian metrik bercampur; Hasil darab Cartesian antara kitaran dan laluan

1. Introduction

Given a simple and connected graph $G = (V, E)$ and two vertices $x, y \in V$, the distance $d_G(x, y)$ (or $d(x, y)$ for short) between x and y is the length of a shortest x - y path. A vertex $v \in V$ is said to distinguish (we also use the terms “recognise” or “determine” instead of “distinguish”) two vertices x and y if $d_G(v, x) \neq d_G(v, y)$. A set $S \subseteq V$ is called a metric generator for G if any pair of distinct vertices of G is distinguished by some element of S . A metric generator of minimum cardinality is called a metric basis, and its cardinality is the metric dimension of G , denoted by $\dim(G)$. The concept of metric dimension was introduced by Slater (1975), where the metric generators were called locating sets, in connection with the problem of uniquely recognising the position of intruders in networks. On the other hand, the concept of metric dimension of a graph was independently introduced by Harary and Melter (1976).

Given a connected graph $G = (V, E)$, a vertex $v \in V$, and an edge $e = uv \in E$, the distance between the vertex v and the edge e is defined as $d_G(e, v) = \min\{d_G(u, v), d_G(w, v)\}$. A vertex $x \in V$ distinguishes (recognises or determines) two edges $e_1, e_2 \in E$ if $d_G(x, e_1) \neq d_G(x, e_2)$. A set S of vertices in a connected graph G is an edge metric generator for G if every two distinct edges of G are distinguished by some vertex of S . The smallest cardinality of an edge metric generator for G is called the edge metric dimension and is denoted by $\text{edim}(G)$. An edge metric basis for G is an edge metric generator for G of cardinality $\text{edim}(G)$.

A vertex $w \in V$ distinguishes two elements (vertices or edges) $x, y \in E \cup V$ if $d_G(w, x) \neq d_G(w, y)$. A set S of vertices in a connected graph G is a mixed metric generator for G if every two distinct elements (vertices or edges) of G are distinguished by some vertex of S . The smallest cardinality of a mixed metric generator for G is called the *mixed metric dimension* and is denoted by $\text{mdim}(G)$ (see (Kelenc *et al.* 2017)).

It follows immediately that any mixed metric generator is also a metric generator and an edge metric generator. In this sense, the following relationship immediately follows. For any graph G ,

$$\text{mdim}(G) \geq \max\{\dim(G), \text{edim}(G)\}.$$

On the other hand, it is not difficult to see that the whole vertex set of any graph G forms a mixed metric generator. Also, any vertex of G and any edge incident to it have the same distance to the vertex itself. In this sense, a vertex alone cannot form a mixed metric generator in G . As a consequence of these observations, the following remark is readily seen to be true.

Remark 1.1. (Kelenc *et al.* 2017) For any graph G of order n , $2 \leq \text{mdim}(G) \leq n$.

Theorem 1.2. (Kelenc *et al.* 2017) Let G be any graph of order n . Then $\text{mdim}(G) = 2$ if and only if G is a path.

Studies of graph products have been intensively conducted over the last few decades, and by now, a rich theory involving the structure and recognition of classes of these graphs has emerged. The most interesting and widely studied graph products are the Cartesian product, the strong product, the direct product, and the lexicographic product, which are also called standard products. Some other graph operations that have attracted several researchers are the corona product, the join graph, the rooted product, and the hierarchical product.

The Cartesian product of the graphs G and H , denoted by $G \square H$, is the graph with the vertex set $V(G) \times V(H) = \{(a, v) : a \in V(G), v \in V(H)\}$, where (a, v) is adjacent to (b, w) whenever $a = b$ and $\{v, w\} \in E(H)$, or $v = w$ and $\{a, b\} \in E(G)$. Where there is no confusion, the vertex (a, v) of $G \square H$ will be written as av .

The edge metric dimension was computed for grid graphs (the Cartesian product of two paths) and for specific cases of torus graphs (the Cartesian product of two cycles) by Kelenc *et al.* (2018). Peterin and Yero (2020) investigated the edge metric dimension of the join, lexicographic, and corona products of graphs. Chandran and Reji (2024) determined the edge metric dimension of the Cartesian product of several graph pairs, including a graph G with K_2 , a cycle with a path, and two cycles. Caceres *et al.* (2007) established bounds for the Cartesian product of graphs G and H by utilising the order of doubly resolving sets. The mixed metric dimension of grid graphs was studied by Kelenc *et al.* (2017), while the mixed metric dimension of the corona product was established by Korivand *et al.* (2024). For further studies on mixed metric dimension, refer to (Alrowaili *et al.* 2022; Danasa *et al.* 2021; Koam *et al.* 2023; Liu *et al.* 2024; Raza *et al.* 2020; Raza & Ji 2020; Rafique *et al.* 2025; Sedlar & Skrekovsk 2021). In this paper, we aim to contribute to the study of mixed metric dimensions by focusing on the Cartesian product of a cycle and a path, commonly referred to as a cylinder grid graph or a generalised prism.

2. Main Results

We now present our main result.

Theorem 2.1. *Let $C_m \square P_n$ be the cylinder grid graph, where $m \geq 4$, m is even, and $n \geq 2$. Then $\text{mdim}(C_m \square P_n) = 3$.*

Proof. Let $G = C_m \square P_n$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_{mn}\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_{2mn-m-n}\}$. We shall show that the set $S = \{v_1, v_{2+m(n-1)}, v_{2+m(n-1)+\frac{m}{2}}\} \subseteq V(G)$ is a mixed metric basis for G .

The representations of the vertices and edges of $G = C_m \square P_n$, indexed by $1 \leq k \leq m$ and $0 \leq j \leq n-1$, are given by

$$r(v_{mi+k} \mid S) = \begin{cases} (j, n-j, \frac{m}{2} + n - 2 - j), & \text{if } k = 1, \\ (1+j, n-1-j, \frac{m}{2} + n - 1 - j), & \text{if } k = 2, \\ (2+j, n-j, \frac{m}{2} + n - 2 - j), & \text{if } k = 3, \\ \vdots \\ (\frac{m}{2} + j, \frac{m}{2} + n - 2 - j, \frac{m}{2} + n - 3 - j), & \text{if } k = \frac{m}{2} + 1, \\ (\frac{m}{2} - 1 + j, \frac{m}{2} + n - 1 - j, \frac{m}{2} + n - 2 - j), & \text{if } k = \frac{m}{2} + 2, \\ (\frac{m}{2} - 2 + j, \frac{m}{2} + n - 2 - j, \frac{m}{2} + n - 1 - j), & \text{if } k = \frac{m}{2} + 3, \\ \vdots \\ (1+j, n+1-j, \frac{m}{2} + n - 2 - j), & \text{if } k = m. \end{cases}$$

We next compute the mixed metric representation of each edge for $1 \leq k \leq m$ and $0 \leq j \leq n-1$:

$$r(e_{mi-m+k} \mid S) = \begin{cases} (j, n-1-j, \frac{m}{2} + n - 3 - j), & \text{if } k = 1, \\ (1+j, n-2-j, \frac{m}{2} + n - 2 - j), & \text{if } k = 2, \\ (2+j, n-1-j, \frac{m}{2} + n - 3 - j), & \text{if } k = 3, \\ \vdots \\ (\frac{m}{2} + j, \frac{m}{2} + n - 3 - j, \frac{m}{2} + n - 1 - j), & \text{if } k = \frac{m}{2} + 1, \\ (\frac{m}{2} - 1 + j, \frac{m}{2} + n - 2 - j, \frac{m}{2} + n - 2 - j), & \text{if } k = \frac{m}{2} + 2, \\ (\frac{m}{2} - 2 + j, \frac{m}{2} + n - 3 - j, \frac{m}{2} + n - 1 - j), & \text{if } k = \frac{m}{2} + 3, \\ (\frac{m}{2} - 3 + j, \frac{m}{2} + n - 4 - j, \frac{m}{2} + n - 1 - j), & \text{if } k = \frac{m}{2} + 4, \\ \vdots \\ (1+j, \frac{m}{2} + n - 3 - j, \frac{m}{2} + n - 4 - j), & \text{if } k = m. \end{cases}$$

For the remaining edges, where $1 \leq k \leq m$ and $0 \leq j \leq n-1$, we have

$$r(e_{m(n-1)+mi+k} \mid S) = \begin{cases} (j, n-1-j, \frac{m}{2} + n-2-j), & \text{if } k=1, \\ (1+j, n-1-j, \frac{m}{2} + n-2-j), & \text{if } k=2, \\ (2+j, n-j, \frac{m}{2} + n-3-j), & \text{if } k=3, \\ \vdots \\ (\frac{m}{2}-1+j, \frac{m}{2} + n-3-j, n-j), & \text{if } k=\frac{m}{2}, \\ (\frac{m}{2}-1+j, \frac{m}{2} + n-2-j, n-1-j), & \text{if } k=\frac{m}{2}+1, \\ (\frac{m}{2}-2+j, \frac{m}{2} + n-2-j, n-1-j), & \text{if } k=\frac{m}{2}+2, \\ (\frac{m}{2}-3+j, \frac{m}{2} + n-3-j, n-j), & \text{if } k=\frac{m}{2}+3, \\ \vdots \\ (j, n-j, \frac{m}{2} + n-3-j), & \text{if } k=m. \end{cases}$$

Note that, from the above representations, no two vertices or edges have the same representation with respect to the resolving set S . Thus, the set $S = \{v_1, v_{2+m(n-1)}, v_{2+m(n-1)+\frac{m}{2}}\}$ uniquely resolves all vertices and edges of $C_m \square P_n$. Hence,

$$\text{mdim}(C_m \square P_n) \leq 3. \quad (1)$$

We now establish that $\text{mdim}(C_m \square P_n) \geq 3$. Since $C_m \square P_n$ is not a path graph, Theorem 1.2 implies that $\text{mdim}(C_m \square P_n) \neq 2$. Moreover, the mixed metric dimension of any graph is at least 2. Therefore,

$$\text{mdim}(C_m \square P_n) \geq 3. \quad (2)$$

Combining the two inequalities, we conclude that

$$\text{mdim}(C_m \square P_n) = 3. \quad \square$$

Therefore, by inequalities (1) and (2), we conclude that $\text{mdim}(C_m \square P_n) = 3$. Theorem 2.1 is illustrated by the following example.

Example 2.2. The mixed metric dimension of $C_6 \square P_3$ is 3; that is, $\text{mdim}(C_6 \square P_3) = 3$.

Let $G = C_6 \square P_3$ be a graph with vertices $V(G) = \{v_1, v_2, v_3, \dots, v_{18}\}$ and edges $E(G) = \{e_1, e_2, e_3, \dots, e_{30}\}$; see Figure 1. We can verify that $S = \{v_1, v_{14}, v_{17}\} \subseteq V(G)$ is a mixed metric generator of $C_6 \square P_3$ with cardinality 3. It can be observed that S resolves the vertices and edges such that no two elements have the same representation. Hence, $\text{mdim}(C_6 \square P_3) \leq 3$. Table 1 presents the metric representations of the vertices and edges of $C_6 \square P_3$. By Theorem 1.2, $\text{mdim}(C_6 \square P_3) \geq 3$. Therefore,

$$\text{mdim}(C_6 \square P_3) = 3.$$

We now present our next theorem.

Theorem 2.3. Let $C_m \square P_n$ be the cylinder grid graph, where $m \geq 5$, m is odd, and $n \geq 2$. Then $\text{mdim}(C_m \square P_n) = 4$.

Proof. Let $G = C_m \square P_n$ and $S = \{v_1, v_{2+m(n-1)}, v_{2+m(n-1)+\frac{m-1}{2}}, v_{3+m(n-1)+\frac{m-1}{2}}\} \subseteq V(G)$. We shall show that S is a mixed metric basis for G .

The representation of each vertex of $G = C_m \square P_n$, for $1 \leq k \leq m$ and $0 \leq j \leq n-1$, is given by

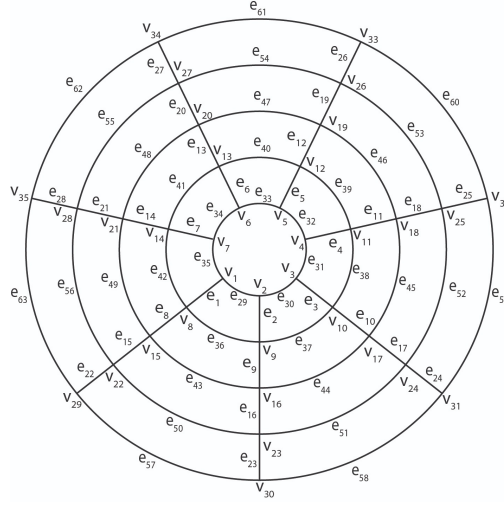

 Figure 1: Graph $C_6 \square P_3$.

 Table 1: Metric coordinates for vertices and edges of $C_6 \square P_3$.

Vertices	Metric Code
v_1	(0, 3, 4)
v_2	(1, 2, 5)
v_3	(2, 3, 4)
v_4	(3, 4, 3)
v_5	(2, 5, 2)
v_6	(1, 3, 3)
v_7	(1, 2, 3)
v_8	(2, 1, 4)
v_9	(3, 2, 3)
v_{10}	(4, 3, 2)
v_{11}	(3, 4, 1)
v_{12}	(2, 2, 2)
v_{13}	(2, 1, 2)
v_{14}	(3, 0, 3)
v_{15}	(4, 1, 2)
v_{16}	(5, 3, 1)
v_{17}	(4, 3, 0)
v_{18}	(3, 1, 1)

Edges	Metric Code
e_1	(0, 2, 3)
e_2	(1, 1, 4)
e_3	(2, 2, 3)
e_4	(3, 3, 2)
e_5	(2, 4, 1)
e_6	(1, 3, 2)
e_7	(1, 1, 2)
e_8	(2, 0, 3)
e_9	(3, 1, 2)
e_{10}	(4, 1, 2)
e_{11}	(3, 3, 0)
e_{12}	(2, 2, 1)
e_{13}	(0, 2, 4)
e_{14}	(1, 2, 4)
e_{15}	(2, 3, 3)

Edges	Metric Code
e_{16}	(2, 4, 2)
e_{17}	(1, 4, 2)
e_{18}	(0, 3, 3)
e_{19}	(1, 1, 3)
e_{20}	(2, 1, 3)
e_{21}	(3, 2, 2)
e_{22}	(3, 3, 1)
e_{23}	(2, 3, 1)
e_{24}	(1, 2, 2)
e_{25}	(2, 0, 2)
e_{26}	(3, 0, 2)
e_{27}	(4, 1, 1)
e_{28}	(4, 2, 0)
e_{29}	(3, 2, 0)
e_{30}	(2, 1, 1)

$$r(v_{mi+k} | S) = \begin{cases} (j, n-j, \frac{m+1}{2} + n - 2 - j, \frac{m+1}{2} + n - 3 - j), & \text{if } k = 1, \\ (1+j, n-1-j, \frac{m+1}{2} + n - 2 - j, \frac{m+1}{2} + n - 2 - j), & \text{if } k = 2, \\ (2+j, n-j, \frac{m+1}{2} + n - 3 - j, \frac{m+1}{2} + n - 2 - j), & \text{if } k = 3, \\ \vdots \\ (\frac{m-1}{2} + j, \frac{m-1}{2} + n - 2 - j, n-j, n+1-j), & \text{if } k = \frac{m+1}{2}, \\ (\frac{m-1}{2} + j, \frac{m-1}{2} + n - 1 - j, n-1-j, n-j), & \text{if } k = \frac{m+1}{2} + 1, \\ (\frac{m-1}{2} - 1 + j, \frac{m-1}{2} + n - 1 - j, n-j, n-1-j), & \text{if } k = \frac{m+1}{2} + 2, \\ (\frac{m-1}{2} - 2 + j, \frac{m-1}{2} + n - 2 - j, n+1-j, n-j), & \text{if } k = \frac{m+1}{2} + 3, \\ \vdots \\ (1+j, n+1-j, \frac{m-1}{2} + n - 2 - j, \frac{m-1}{2} + n - 3 - j), & \text{if } k = m. \end{cases}$$

We next compute the mixed metric representation of each edge, for $1 \leq k \leq m$ and $0 \leq j \leq n-1$:

$$r(e_{mi-m+k} \mid S) = \begin{cases} (j, n-1-j, \frac{m-1}{2} + n-2-j, \frac{m-1}{2} + n-3-j), & \text{if } k=1, \\ (1+j, n-2-j, \frac{m-1}{2} + n-2-j, \frac{m-1}{2} + n-2-j), & \text{if } k=2, \\ (2+j, n-1-j, \frac{m-1}{2} + n-3-j, \frac{m-1}{2} + n-2-j), & \text{if } k=3, \\ \vdots \\ (\frac{m-1}{2} + j, \frac{m-1}{2} + n-1-j, n-1-j, n-2-j), & \text{if } k = \frac{m+1}{2}, \\ (\frac{m-1}{2} + j, \frac{m-1}{2} + n-2-j, n-2-j, n-3-j), & \text{if } k = \frac{m+1}{2} + 1, \\ (\frac{m-1}{2} - 1 + j, \frac{m-1}{2} + n-2-j, n-3-j, n-2-j), & \text{if } k = \frac{m+1}{2} + 2, \\ \vdots \\ (1+j, n-j, \frac{m-1}{2} + n-3-j, \frac{m-1}{2} + n-4-j), & \text{if } k=m. \end{cases}$$

For the remaining edges, where $1 \leq k \leq m$ and $0 \leq j \leq n-1$, we have

$$r(e_{m(n-1)+mi+k} \mid S) = \begin{cases} (j, n-1-j, \frac{m+1}{2} + n-2-j, \frac{m+1}{2} + n-3-j), & \text{if } k=1, \\ (1+j, n-1-j, \frac{m+1}{2} + n-3-j, \frac{m+1}{2} + n-2-j), & \text{if } k=2, \\ (2+j, n-j, \frac{m+1}{2} + n-4-j, \frac{m+1}{2} + n-1-j), & \text{if } k=3, \\ \vdots \\ (\frac{m-1}{2} + j, \frac{m-1}{2} + n-2-j, n-1-j, n-j), & \text{if } k = \frac{m+1}{2}, \\ (\frac{m-1}{2} - 1 + j, \frac{m-1}{2} + n-1-j, n-1-j, n-1-j), & \text{if } k = \frac{m+1}{2} + 1, \\ (\frac{m-1}{2} - 2 + j, \frac{m-1}{2} + n-2-j, n-j, n-1-j), & \text{if } k = \frac{m+1}{2} + 2, \\ (\frac{m-1}{2} - 3 + j, \frac{m-1}{2} + n-3-j, n+1-j, n-j), & \text{if } k = \frac{m+1}{2} + 3, \\ \vdots \\ (j, n-j, \frac{m-1}{2} + n-2-j, \frac{m-1}{2} + n-3-j), & \text{if } k=m. \end{cases}$$

Note that, from the above representations, no two vertices or edges have the same representation with respect to the resolving set S . Thus, the set $S = \{v_1, v_{2+m(n-1)}, v_{2+m(n-1)+\frac{m-1}{2}}, v_{3+m(n-1)+\frac{m-1}{2}}\}$ uniquely resolves the vertices and edges of G . Hence,

$$\text{mdim}(C_m \square P_n) \leq 4. \quad (3)$$

Conversely, we need to show that $\text{mdim}(C_m \square P_n) \geq 4$. By Theorem 1.2, we know that $\text{mdim}(C_m \square P_n) \neq 2$. We proceed by contradiction and suppose that $\text{mdim}(C_m \square P_n) = 3$. To this end, we consider the following three cases.

Case 1. Suppose that all three resolving vertices are taken from the first cycle $C_m^{(1)}$. We consider the following subcases.

Subcase 1.1. Suppose that the resolving vertices are selected from the set $\{v_i : 1 \leq i \leq m\}$. Without loss of generality, assume that the first resolving vertex is v_1 and the other two resolving vertices are v_j and v_s , where $2 \leq j, s \leq \frac{m+1}{2}$ and $v_j \neq v_s$. Then $d(v_k \mid v_1, v_j, v_s) = d(e_k \mid v_1, v_j, v_s)$, where $1 \leq k \leq \frac{m+1}{2}$. For the remaining choices of v_j and v_s , where $\frac{m+1}{2} \leq j, s \leq m-1$, the cases are symmetric to those for $2 \leq j, s \leq \frac{m+1}{2}$ and hence are identical.

Subcase 1.2. When all three resolving vertices are taken from the second cycle, the argument is identical to that in Subcase 1.1 due to the symmetry of the graph.

Case 2. Suppose that one resolving vertex is taken from the first cycle $C_m^{(1)}$ and two resolving vertices are taken from $C_m^{(2)}$. Assume that the first resolving vertex is v_1 and the other two resolving vertices are v_j and v_s , where $m+1 \leq j, s \leq \frac{3m+1}{2}$. Then $d(v_k | v_1, v_j, v_s) = d(e_{(n-1)m+k+1} | v_1, v_j, v_s)$, where $1 \leq k \leq \frac{m-1}{2} + 1$. For the remaining choices of vertices from $C_m^{(2)}$, the cases are identical due to the symmetry of the graph.

Subcase 2.1. If the first resolving vertex is taken from the second cycle and the other two resolving vertices are taken from the third cycle, the argument is identical to that in Subcase 1.2 due to the symmetry of the graph.

Subcase 2.2. If the first resolving vertex is taken from the first cycle and the other two resolving vertices are taken from the third cycle, the argument is identical to that in Subcase 2.1.

Case 3. Suppose that one resolving vertex is taken from each of $C_m^{(1)}$, $C_m^{(2)}$, and $C_m^{(3)}$. Assume that the first resolving vertex is v_1 , the second resolving vertex is v_j , where $m+1 \leq j \leq \frac{3m+1}{2}$, and the third resolving vertex is v_s , where $2m+1 \leq s \leq \frac{5m+1}{2}$. Then $d(v_k | v_1, v_j, v_s) = d(e_{(n-1)m+k+1} | v_1, v_j, v_s)$, where $1 \leq k \leq \frac{m-1}{2} + 1$.

When one resolving vertex is taken from each of $C_m^{(2)}$, $C_m^{(3)}$, and $C_m^{(4)}$, the argument is identical due to the symmetry of the graph.

From the above cases, we obtain $\text{mdim}(C_m \square P_n) \neq 3$, which contradicts our assumption. Hence,

$$\text{mdim}(C_m \square P_n) \geq 4. \quad (4)$$

Therefore, by inequalities (3) and (4), we conclude that $\text{mdim}(C_m \square P_n) = 4$. $\square$
Theorem 2.3 is illustrated by the following example.

Example 2.4. The mixed metric dimension of $C_7 \square P_5$ is 4; that is, $\text{mdim}(C_7 \square P_5) = 4$.

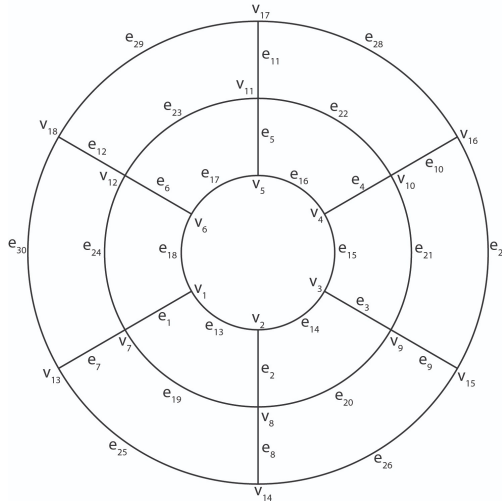


Figure 2: Graph $C_7 \square P_5$.

Let $G = C_7 \square P_5$ be a graph with vertices $V(G) = \{v_1, v_2, v_3, \dots, v_{35}\}$ and edges $E(G) = \{e_1, e_2, e_3, \dots, e_{63}\}$; see Figure 2. We can verify that $S = \{v_1, v_{30}, v_{33}, v_{34}\} \subseteq V(G)$ is a

mixed metric generator of $C_7 \square P_5$ with cardinality 4. It can be observed that S resolves the vertices and edges such that no two elements have the same representation. Hence, $\text{mdim}(C_7 \square P_5) \leq 4$. Table 2 presents the metric representations of the vertices and edges of $C_7 \square P_5$. By the same argument as in the proof of Theorem 2.3, $\text{mdim}(C_7 \square P_5) \geq 4$. Therefore, $\text{mdim}(C_7 \square P_5) = 4$.

Table 2: Metric coordinates for vertices and edges of $C_7 \square P_5$.

Vertices	Metric Code	Edges	Metric Code	Edges	Metric Code
v_1	(0, 5, 7, 6)	e_1	(0, 4, 6, 5)	e_{33}	(2, 7, 4, 4)
v_2	(1, 4, 7, 7)	e_2	(1, 3, 6, 6)	e_{34}	(1, 6, 5, 4)
v_3	(2, 5, 6, 7)	e_3	(2, 4, 5, 6)	e_{35}	(0, 5, 6, 5)
v_4	(3, 6, 5, 6)	e_4	(3, 5, 4, 5)	e_{36}	(1, 3, 6, 5)
v_5	(3, 7, 4, 5)	e_5	(3, 6, 3, 4)	e_{37}	(2, 3, 5, 6)
v_6	(2, 7, 5, 4)	e_6	(2, 6, 4, 3)	e_{38}	(3, 4, 4, 5)
v_7	(1, 6, 6, 5)	e_7	(1, 5, 5, 4)	e_{39}	(4, 5, 3, 4)
v_8	(1, 4, 6, 5)	e_8	(1, 3, 5, 4)	e_{40}	(3, 6, 3, 3)
v_9	(2, 3, 6, 6)	e_9	(2, 2, 5, 5)	e_{41}	(2, 5, 4, 3)
v_{10}	(3, 4, 5, 6)	e_{10}	(3, 3, 4, 5)	e_{42}	(1, 4, 5, 4)
v_{11}	(4, 5, 4, 5)	e_{11}	(4, 4, 3, 4)	e_{43}	(2, 2, 5, 4)
v_{12}	(4, 6, 3, 4)	e_{12}	(4, 5, 2, 3)	e_{44}	(3, 2, 4, 5)
v_{13}	(3, 6, 4, 3)	e_{13}	(3, 5, 3, 2)	e_{45}	(4, 3, 3, 4)
v_{14}	(2, 5, 5, 4)	e_{14}	(2, 4, 4, 3)	e_{46}	(5, 4, 2, 3)
v_{15}	(2, 3, 5, 4)	e_{15}	(2, 2, 4, 3)	e_{47}	(4, 5, 2, 2)
v_{16}	(3, 2, 5, 5)	e_{16}	(3, 1, 4, 4)	e_{48}	(3, 4, 3, 2)
v_{17}	(4, 3, 4, 5)	e_{17}	(4, 2, 3, 4)	e_{49}	(2, 3, 4, 3)
v_{18}	(5, 4, 3, 4)	e_{18}	(5, 3, 2, 3)	e_{50}	(3, 1, 4, 3)
v_{19}	(5, 5, 2, 3)	e_{19}	(5, 4, 1, 2)	e_{51}	(4, 1, 3, 4)
v_{20}	(4, 5, 3, 2)	e_{20}	(4, 4, 2, 1)	e_{52}	(5, 2, 2, 3)
v_{21}	(3, 4, 4, 3)	e_{21}	(3, 3, 3, 2)	e_{53}	(6, 3, 1, 2)
v_{22}	(3, 2, 4, 3)	e_{22}	(3, 1, 3, 2)	e_{54}	(5, 4, 1, 1)
v_{23}	(4, 1, 4, 4)	e_{23}	(4, 0, 3, 3)	e_{55}	(4, 3, 2, 1)
v_{24}	(5, 2, 3, 4)	e_{24}	(5, 1, 2, 3)	e_{56}	(3, 2, 3, 2)
v_{25}	(6, 3, 2, 3)	e_{25}	(6, 2, 1, 2)	e_{57}	(4, 0, 3, 2)
v_{26}	(6, 4, 1, 2)	e_{26}	(6, 3, 0, 1)	e_{58}	(5, 0, 2, 3)
v_{27}	(5, 4, 2, 1)	e_{27}	(5, 3, 1, 0)	e_{59}	(6, 1, 1, 2)
v_{28}	(4, 3, 3, 2)	e_{28}	(4, 2, 2, 1)	e_{60}	(7, 2, 0, 1)
v_{29}	(4, 1, 3, 2)	e_{29}	(0, 4, 7, 6)	e_{61}	(6, 3, 0, 0)
v_{30}	(5, 0, 3, 3)	e_{30}	(1, 4, 6, 7)	e_{62}	(5, 2, 1, 0)
v_{31}	(6, 1, 2, 3)	e_{31}	(2, 5, 5, 6)	e_{63}	(4, 1, 2, 1)
v_{32}	(7, 2, 1, 2)	e_{32}	(3, 6, 4, 5)		
v_{33}	(7, 3, 0, 1)				
v_{34}	(6, 3, 1, 0)				
v_{35}	(5, 2, 2, 1)				

3. Conclusions

In this research, we have explored the mixed metric dimension of the Cartesian product of a cycle C_m and a path P_n . By analysing the structural properties of these graph products, we established exact values for their mixed metric dimensions, as presented in Theorems 2.1 and 2.3. These results contribute to a broader understanding of how graph operations influence the cardinality of minimal resolving sets when both vertices and edges must be uniquely identified. While our findings provide a characterisation of the mixed metric dimension of $C_m \square P_n$, several directions for further investigation remain.

a) Mixed metric dimension of cycle products

A natural extension of this work is to determine the exact values of the mixed metric dimension

for the Cartesian product of two cycles, $C_n \square C_m$. The symmetry and absence of end vertices in these graphs, unlike Cartesian products involving paths, may present additional challenges in determining minimal resolving sets.

b) Other graph products

Future studies may investigate the behaviour of the mixed metric dimension under other fundamental graph operations, such as the strong, lexicographic, and direct products. Each of these products alters the adjacency and distance properties in different ways, potentially leading to different behaviour of the mixed metric dimension.

c) General bounds for product graphs

Beyond specific graph families, further research may focus on developing tighter general upper and lower bounds for the mixed metric dimension of the Cartesian product $G \square H$ in terms of the corresponding properties of G and H .

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