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ADAPTIVE PREDICTOR-CORRECTOR SCHEMES FOR A CLASS OF PANTOGRAPH-TYPE DELAY DIFFERENTIAL EQUATIONS

(*Skim Peramal-Pembetul Penyesuaian untuk Kelas Persamaan Pembezaan Lengah Jenis Pantograf*)

NUR AISHAH JAMALUDIN & NURUL HUDA ABDUL AZIZ*

ABSTRACT

This study highlights the new adaptive predictor-corrector scheme which is combined with the Gauss' forward interpolation strategy to handle the properties of time delays in a single delay pantograph. This scheme provides an entirely new perspective on accurate and efficient approaches since the scheme considered a technique that employs both odd and even differences under the central interpolation approximation which can be adapted in the particular pantograph delays. To guarantee that the time delay approximation yields a high degree of accuracy in comparison to alternative methods, a thorough analysis is given to the interpolation node selection. Aside from that, the convergence and stability analysis which includes the order, error constant, zero-stability and consistency of the suggested approach are examined. Numerical problems are provided to show the practical advantages of the suggested algorithms and to compare with the exact results. Based on the findings, it can be concluded that the numerical method of the adaptive predictor-corrector scheme, which was used to solve the single pantograph delay, is reliable with high error accuracy. The novelty of the method lies in its adaptive multistep structure combined with Gauss's interpolation, which enhances precision and improves the handling of time-delay terms.

Keywords: vanishing lags; stability analysis; proportional delays; Gauss's forward interpolation

ABSTRAK

Kajian ini menekankan mengenai skema peramal-pembetul penyesuaian baharu yang digabungkan dengan strategi interpolasi ke hadapan Gauss untuk mengendalikan sifat lengah masa dalam pantograf lengah tunggal. Skim ini memberikan perspektif yang sama sekali baharu tentang pendekatan yang tepat dan cekap kerana skim ini dianggap sebagai teknik yang menggunakan kedua-dua perbezaan ganjil dan genap di bawah penghampiran interpolasi pusat yang boleh disesuaikan dalam lengah pantograf tertentu. Untuk menjamin bahawa anggaran lengah masa menghasilkan ketepatan yang tinggi berbanding kaedah alternatif, analisis menyeluruh diberikan kepada pemilihan nod interpolasi. Selain itu, analisis penumpuan dan kestabilan yang merangkumi susunan, pemalar ralat, kestabilan sifar dan ketekalan pendekatan yang dicadangkan dinilai. Masalah berangka disediakan untuk menunjukkan kelebihan praktikal algoritma yang dicadangkan dan untuk membandingkan dengan keputusan yang tepat. Berdasarkan kajian, dapat disimpulkan bahawa kaedah berangka skema peramal-pembetul penyesuaian, yang digunakan untuk menyelesaikan lengah pantograf tunggal, boleh dipercayai dengan ketepatan ralat yang tinggi. Keistimewaan kaedah ini terletak pada struktur penyesuaian berbilang langkah bersama interpolasi Gauss, yang meningkatkan ketepatan dan keupayaan menangani terma kelewatan dengan lebih berkesan.

Kata kunci: lengah lenyap; analisis kestabilan; lengah berkadar; interpolasi ke hadapan Gauss

1. Introduction

A differential equation comprises of one or more functions associated with their derivatives. They are well-known to serve as basic mathematical framework for simulating a broad spectrum of scientific, engineering, and physical phenomena. Delay differential equations (DDEs) are one type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at a previous time (Jamilla *et al.* 2020). Since the solution of DDEs requires knowledge of the current state and certain previous time states, therefore DDEs has comprehensive applications in explaining different phenomena in various fields, such as sciences, engineering, and economics, according to (Buedo & Liz 2018; Rihan 2021; Sardar *et al.* 2021).

Ordinary differential equations (ODEs) and DDEs are two equations of describing physical phenomena, but they differ in that ODEs' derivatives of unknown functions rely solely on the present value of the independent variable. The features, behavior, and solutions of delay differential equations have thus far been the subject of extensive research by numerous scholars (Aziz *et al.* 2023; Chen *et al.* 2020; Ockendon & Tayler 1971; Zhang *et al.* 2021). This is because DDEs prove to be essential in situations when ODE-based models fail.

Based on Ockendon and Tayler (1971), to model the movement of the pantograph head on an electric locomotive, they suggested a first-order pantograph delay differential equation, which is provided by

$$u'(t) = au(t) + bu(ct), \qquad t > 0, \tag{1}$$

where a and b are real constants and c is a pantograph time delay in the range, $0 < c < 1$. Eq. (1) is subjected to the initial condition

$$u(0) = \lambda, \tag{2}$$

where λ is a real constant.

Note that the proportional delay is part of the pantograph time delay equation. The numerical solution of this equation is more challenging when the delays are variable or proportional than when they are constant. A number of scenarios will arise when working with proportional delay, such as vanishing lag and circumstances in which the delay term is inside the range of preceding points. In order to address with all the properties of the time delays, a thorough localization technique must be taken into consideration.

Several scholars have conducted in-depth analytical and numerical solutions of pantograph time delay. To name a few, mono-implicit Runge Kutta order two (Rihan 2024), discontinuous Galerkin method (Jiang *et al.* 2020), Legendre pseudospectral method (Jafari *et al.* 2021) and the homotopy pertubation method in (Albidah *et al.* 2023; Vilu *et al.* 2023). In the study by Bahgat (2020), the author approximates an analytical solution for the multi-pantograph delay with higher-order differential equations by combining the Laplace transform with the variational iteration method. In addition to that, initial value problems of linear and non-linear pantograph delay differential equations were solved using feedforward artificial neural networks with Levenberg-Marquardt backpropagation and Bayesian regularization in (Khan *et al.* 2020).

Meanwhile, some of the researchers have discovered few studies in the predictor-corrector scheme such as (Li *et al.* 2011) solving the nonlinear parabolic differential equations, (Jayakumar & Kanagarajan 2014) investigating the hybrid fuzzy differential equations, and (Oghonyon 2018) solving the fourth order ordinary differential equation in the block method. Moreover, a strong predictor-corrector approach was also developed for numerical solutions of Ito-type stochastic delay differential equations in (Niu *et al.* 2015). A sixth-order Adams-Bashforth-Moulton block method was also introduced to solve constant and time-dependent neutral delay equations with better accuracy (Puzi & Aziz 2023). Also, the author in (Kumar & Daftardar-Gejji 2019) suggested a new family of six predictor-corrector methods for resolving fractional differential equations that are not linear. A similar application showed how numerical schemes can be tailored to problems with uncertainty by proposing a fuzzy numerical method for addressing a one-dimensional steady-state heat conduction issue with a constant gradient (Husin *et al*. 2025).

Despite the existence of various numerical schemes for solving pantograph-type delay differential equations, many of these approaches either assume constant delays or require complex implementation structures, particularly when dealing with proportional or vanishing lags. Most existing predictor-corrector methods in the literature do not adequately address the challenges of delay term approximation when the delay lies between known mesh points, especially in single-delay pantograph models.

Motivated by the limitations of existing approaches, this study introduces an adaptive predictor-corrector (APC) scheme that uses Gauss's forward interpolation to better approximate the delayed term. What sets this method apart is the way it combines the structure of the APC scheme with Gauss's forward interpolation to solve the single-pantograph time-delay problem described in Eq. (1). Furthermore, the paper provides a thorough analysis of the convergence and stability of the method.

The following is the structure of the paper. The mathematical formulation of the suggested APC scheme is explained in Section 2, which is followed by both convergence and stability analysis. Sections 3 and 4 respectively discuss the application of Gauss's forward interpolation in addressing the characteristics of single-pantograph time delays, and present a systematic computational algorithm of the APC scheme designed to solve pantograph delay differential equations. Next, the applicability and efficiency of the method are tested through some numerical problems in Section 5. In Section 6, the findings are further examined, and in Section 7, the study is finally concluded.

2. Numerical Method of Adaptive Predictor-Corrector Scheme

The APC scheme, which was developed using Newton's forward interpolation and modified for proportional delay terms in pantograph equations, is introduced in this section.

2.1. *Formulation of APC scheme*

Let $[a, b]$ be the interval over which the solution of initial value problem

$$\frac{du}{dt} = f(t, u), \qquad u(t_0) = u_0. \tag{3}$$

For this, the interval $[a, b]$ is divided into n equal sub-intervals such that

$$t_i = t_0 + ih, i = 1, 2, 3, ..., n, \tag{4}$$

where h is the step size. The Newton's forward interpolation formula is

$$u_p = u_0 + p\Delta u_0 + \frac{p(p-1)}{2!}\Delta^2 u_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 u_0 + ..., \tag{5}$$

where $p = \frac{t-t_0}{h}$. For $u' = f(t, u)$, the above formula takes the following form

$$u' = u'_0 + p\Delta u'_0 + \frac{p(p-1)}{2!}\Delta^2 u'_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 u'_0 + ... \tag{6}$$

To identify $u_4 = u(t_0 + 4h)$, Eq. (6) can be integrated with respect to t as follows

$$\int_{t_0}^{t_0+4h} u'\,dt = \int_{t_0}^{t_0+4h} \left[u'_0 + p\Delta u'_0 + \frac{p^2-p}{2!}\Delta^2 u'_0 + \frac{p^3-3p^2+2p}{3!}\Delta^3 u'_0 + ...\right] dt. \quad (7)$$

As $p = \frac{t-t_0}{h}$ which implies that $dp = \frac{1}{h}dt$, and $dt = hdp$. Note that, when t approaching to t_0, p becomes 0, and when t approaching to $t_0 + 4h$, p becomes 4. Hence, Eq. (7) has the form:

$$[u]_{t_0}^{t_0+4h} = h\int_0^4 \left[u'_0 + p\Delta u'_0 + \frac{p^2-p}{2!}\Delta^2 u'_0 + \frac{p^3-3p^2+2p}{3!}\Delta^3 u'_0 + ...\right] dp. \quad (8)$$

Then, we obtained

$$u_4 - u_0 = h\left[pu'_0 + \frac{p^2}{2}\Delta u'_0 + \frac{1}{2!}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 u'_0 + \frac{1}{3!}\left(\frac{p^4}{4} - p^3 + p^2\right)\Delta^3 u'_0 + ...\right]_0^4. \quad (9)$$

Thus,

$$u_4 = u_0 + h\left[4u'_0 + 8\Delta u'_0 + \frac{1}{2!}\left(\frac{40}{3}\right)\Delta^2 u'_0 + \frac{8}{3}\Delta^3 u'_0 + ...\right]. \quad (10)$$

Neglecting fourth and higher order differences, then

$$u_4 = u_0 + h\left[4u'_0 + 8(u'_1 - u'_0) + \frac{20}{3}(u'_2 - 2u'_1 + u'_0) + \frac{8}{3}(u'_3 - 3u'_2 + 3u'_1 - u'_0)\right]. \quad (11)$$

By simplifying the equation, then we have

$$u_4 = u_0 + h\left[\frac{8}{3}u'_3 - \frac{4}{3}u'_2 + \frac{8}{3}u'_1\right]. \quad (12)$$

In general, Eq. (12) can be written as

$$u^p_{k+1} = u_{k-3} + \frac{4h}{3}\left[2u'_k - u'_{k-1} + 2u'_{k-2}\right]. \quad (13)$$

The formula in Eq. (13) is known as adaptive predictor formula. Next, the corrector formula is derived similarly to the predictor by first integrating Eq. (5) with respect to t over $[t_0, t_0 + 2h]$,

$$\int_{t_0}^{t_0+2h} u'\,dt = \int_0^2 \left[u'_0 + p\Delta u'_0 + \frac{p^2-p}{2!}\Delta^2 u'_0 + \frac{p^3-3p^2+2p}{3!}\Delta^3 u'_0 + ...\right] dp. \quad (14)$$

Then, by evaluating the integral, gives

$$u_2 - u_0 = h\left[pu'_0 + \frac{p^2}{2}\Delta u'_0 + \frac{1}{2!}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 u'_0 + \frac{1}{3!}\left(\frac{p^4}{4} - p^3 + p^2\right)\Delta^3 u'_0 + ...\right]_0^2. \quad (15)$$

Neglecting fourth and higher order differences, produces

$$u_2 - u_0 = h\left[pu_0' + \frac{p^2}{2}\Delta u_0' + \frac{1}{2!}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 u_0' + \frac{1}{3!}\left(\frac{p^4}{4} - p^3 + p^2\right)\Delta^3 u_0' + ...\right]_0^2. \tag{16}$$

Following that,

$$u_2 - u_0 = h\left[2u_0' + 2\Delta u_0' + \frac{1}{2!}\left(\frac{2}{3}\right)\Delta^2 u_0' + \frac{1}{3!}(0)\Delta^3 u_0'\right]. \tag{17}$$

$$u_2 = u_0 + h\left[2u_0' + 2(u_1' - u_0') + \frac{1}{3}(u_2' - 2u_1' + u_0')\right]. \tag{18}$$

Therefore, when the equation is simplified, we get

$$u_2 = u_0 + \frac{h}{3}[u_2' + 4u_1' + u_0']. \tag{19}$$

In general, Eq. (19) can be written as

$$u_{k+1}^c = u_{k-1} + \frac{h}{3}[u_{k+1}' + 4u_k' + u_{k-1}'], \tag{20}$$

which is known as adaptive corrector formula.

2.2. *Convergence analysis of the method*

In this particular section, the method's order and convergence are studied. The definition provided by Darus *et al.* (2023) was utilized.

Definition 2.1. If $C_0 = C_1 = ... = C_m = 0$ and $C_{m+1} \neq 0$, then the linear multistep technique is considered to be of order m.

Definition 2.2. A linear k-step method is considered consistent if and only if its order of accuracy satisfies $|m| \geq 1$.

Definition 2.3. If multiple zeroes z satisfy $|\rho(z)| < 1$ and the first characteristic polynomial, $|\rho(z)| \leq 1$, then a linear multistep method is considered zero stable. The first characteristic polynomial is defined as

$$\rho(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + ... + \alpha_k z^k, \tag{21}$$

where, α_j are the coefficients of the linear multistep method.

The formula beneath is utilized to identify the method's order

$$\sum_{j=0}^{k}[\alpha_j u(t+jh) - h\beta_j u'(t+jh)] = C_m u^m + O(h^{m+1}), \tag{22}$$

where, m is the order of the linear multistep method and $O(h^{m+1})$ is the local truncation error. The constant C_m can be calculated using the following formula

$$C_m = \sum_{j=0}^{k} \left[\frac{j^m \alpha_j}{m!} - \frac{j^{m-1}\beta_j}{(m-1)!} \right], \quad m = 0, 1, 2, \dots \tag{23}$$

where, k is the step, α and β are the coefficients obtained from the proposed method. C_{m+1} is referred as the error constant of the method. The coefficients is calculated as follows:

$$\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j = 0,\\
C_1 &= \sum_{j=0}^{k} (j\alpha_j - \beta_j) = 0,\\
C_2 &= \sum_{j=0}^{k} \left(\frac{j^2\alpha_j}{2!} - j\beta_j \right) = 0,\\
C_3 &= \sum_{j=0}^{k} \left(\frac{j^3\alpha_j}{3!} - \frac{j^2\beta_j}{2!} \right) = 0,\\
C_4 &= \sum_{j=0}^{k} \left(\frac{j^4\alpha_j}{4!} - \frac{j^3\beta_j}{3!} \right) = 0,\\
C_5 &= \sum_{j=0}^{k} \left(\frac{j^5\alpha_j}{5!} - \frac{j^4\beta_j}{4!} \right) = -\frac{1}{90}.
\end{aligned} \tag{24}$$

Therefore, this method is of order 4 with error constant $C_5 = -\frac{1}{90}$ and also said to be consistent since it satisfies the Definition 3.2.

In order to verify the zero stable features for the suggested approach, which is based on Eq. (20), the characteristic polynomial is

$$\rho(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 = -1 + z^2 = (z+1)(z-1), \tag{25}$$

which has two zeros, $z = -1$ and $z = 1$. These both have magnitude less than or equal to 1 and multiples zeros satisfy the condition in Definition 3.3. Thus, the proposed method is said to be zero stable.

Definition 2.4. The method appears to be convergent if it is zero stable and consistent.

We have demonstrated that the proposed method is zero stable and consistent, hence proving its convergence.

2.3. *Stability of the method*

Stability is a fundamental property of numerical methods for delay differential equations, ensuring that solutions remain bounded under small perturbations (Lambert 1991; Hairer *et al.* 1993). While accuracy concerns the closeness to the exact solution, stability guarantees reliable long-term behavior (Rihan 2021). In delay equations, where current values depend on past states, instability may occur even for small or vanishing lags if the method lacks sufficient stability.

The concept of P-stability, introduced by Lambert (1991), characterizes methods that remain stable for all negative real parameters and any positive delay. Its relevance has been further highlighted in applied settings (Rihan 2021; Aziz *et al*. 2014), particularly for problems involving small or vanishing lags. To analyze the stability of the proposed APC scheme, we consider the standard linear test equation as

$$\begin{aligned} u'(t) &= \lambda u(t) + \mu u(t-\tau), \qquad && t \geq t_0, \\ u(t) &= \phi(t), && -\tau \leq t \leq t_0, \end{aligned} \tag{26}$$

where λ and μ are real numbers, τ is the delay term such as $\tau = rh$ is a constant step size such that $t_n = t_0 + nh$ and $r \in \mathbb{Z}^+$.
Applying Eq. (20) to to this test equation leads to:

$$\begin{aligned} u_{k+1}(t) - u_{k-1}(t) = \frac{h}{3}[\lambda u_{k+1}(t) + \mu u_{k+1}(t-\tau) + 4[\lambda u_k(t) + \mu u_k(t-\tau)] + \\ \lambda u_{k-1}(t) + \mu u_{k-1}(t-\tau)]. \end{aligned} \tag{27}$$

Rearranging Eq. (27) to be equal to zero and let $H_1 = h\lambda$ and $H_2 = h\mu$, then we obtained

$$\begin{aligned} &u_{k+1}(t) - H_1 u_{k+1}(t) - u_{k-1}(t) - \frac{1}{3}H_1 u_{k-1}(t) - \frac{4}{3}H_1 u_k(t) \\ &-\frac{1}{3}H_2 u_{k+1}(t-\tau) - \frac{4}{3}H_2 u_k(t-\tau) - \frac{1}{3}H_2 u_{k-1}(t-\tau) = 0. \end{aligned} \tag{28}$$

From this, the characteristic polynomial $\pi(t)$ is derived as

$$\pi(t) = \left(1 - \frac{H_1}{3}\right) t^{2+m} - \left(1 + \frac{1}{3}H_1\right) t^m - \frac{4}{3}H_1 t^{1+m} - \left(\frac{1}{3}t^2 - \frac{4}{3}t - \frac{1}{3}\right) H_2 = 0. \tag{29}$$

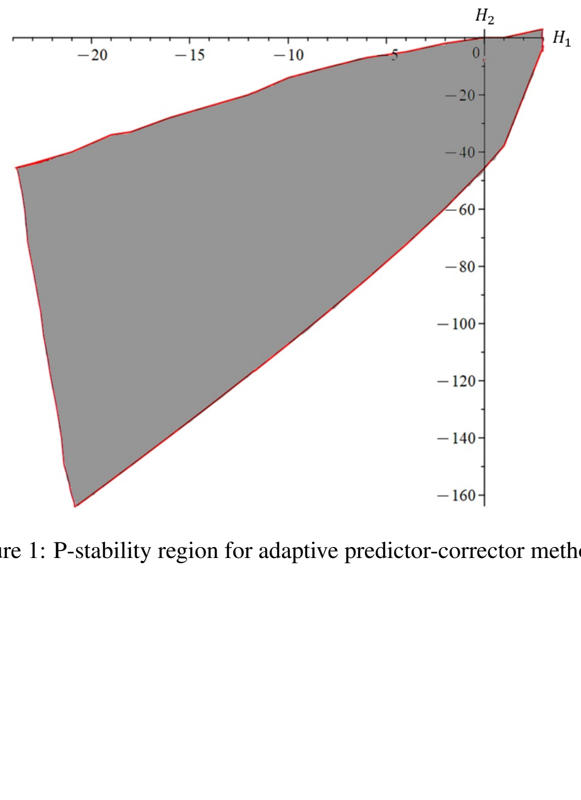

Figure 1: P-stability region for adaptive predictor-corrector method

The P-stability region of the proposed APC scheme in the (H_1, H_2) plane is displayed in Figure 1, where $H_1 = h\lambda$ and $H_2 = h\mu$ represent the scaled values of the non-delay and delay terms, respectively. This region helps us understand where the numerical method stays stable over time. From the figure, we can clearly see that the method is stable for all negative values of H_1 (which corresponds to $\lambda < 0$) and for any positive delay term H_2, which meets the formal definition of P-stability. This is especially important when solving delay differential equations, where delays can easily cause instability. The shape of the stability region is wide which indicates that the method can handle both small and vanishing lags without breaking down or producing erratic results. In other words, the figure confirms that the APC method remains reliable and accurate even in challenging scenarios involving proportional or vanishing lags. This strong and consistent stability makes the method highly suitable for solving pantograph-type delay differential equations.

3. Gauss's Forward Interpolation Strategy

This section explains how the proposed method handles different types of delays that can appear in single-pantograph delay differential equations. In particular, delay values may either fall exactly on a previously computed time point or between two time points. These situations are commonly referred to as *vanishing lag* and *small lag*, respectively, as discussed in (Aziz & Majid 2013).

To accurately approximate delayed terms such as $u(\alpha t)$, especially when the delay does not align with existing mesh points, Gauss's forward interpolation is applied. This interpolation method improves accuracy by using both even and odd forward differences, making it suitable for the proportional and vanishing lags common in pantograph-type equations. Two main cases are considered when estimating the delayed term:

3.1. *Case 1: Delay time fall exactly at* t_n

Definition 3.1. Vanishing lag is the circumstance where the time delay vanishes out because it falls exactly at one of the preceding points.

A vanishing lag occurs when the delay term, αt coincides exactly with one of the computed time steps, such as $\alpha t = t_n$. In this situation, the delayed value $u(\alpha t)$ can be directly substituted using the previously calculated solution value, with no interpolation required. For instance, in Problem 1 of Section 5, we have

$$u'(t) = \frac{1}{2}\exp\left(\frac{1}{2}t\right)u\left(\frac{1}{2}t\right) + \frac{1}{2}u(t), \tag{30}$$

$$u(0) = 1, \qquad 0 \leq t \leq 1. \tag{31}$$

From Eq. (30), it can be seen that there may exist a vanishing lag in the pantograph delay as $(\frac{t}{2}) \to 0$ where the time delay will tend to $u(\frac{1}{2}(0)) = u(0) = 1 = t_0$. Therefore, the technique of allocating the value of earlier phases will be considered in order to handle this characteristic.

3.2. *Case 2: Delay time fall in the range* $[t_{n-1}, t_n]$

In other scenarios, when the delay lies between two known time steps (e.g., $\alpha t \in [t_{n-1}, t_n]$), the Gauss's forward interpolation is used to approximate the delayed value based on nearby known values. The use of up to ten previous points in the interpolation process supports accurate and stable results when solving equations with proportional delay. Consider the following example: Eq. (30) with the current time step $t_9 = 0.009$ for $h = 0.001$, where $u(\frac{1}{2}(0.009))$ has a delay term of $u(0.0045)$. It can be observed that the delay is currently within the range of $[0.004, 0.005]$. Thus, ten discrete data sets will be selected to be applied in the Gauss's forward interpolation formula as

$$\begin{aligned} u_q = u_0 + q\Delta u_0 + \frac{q(q-1)}{2!}.\Delta^2 u_{-1} + \frac{(q+1)q(q-1)}{(3!)}.\Delta^3 u_{-1} \\ + \frac{(q+1)q(q-1)(q-2)}{4!}.\Delta^4 u_{-2} + ..., \end{aligned} \tag{32}$$

where

$$q = \frac{t - t_0}{h}.$$

Incorporated within the APC scheme, this interpolation technique plays an important role in preserving numerical accuracy when solving delay differential equations.

4. Computational Algorithm of APC Scheme

The following algorithm provides an outline of the means by which the APC scheme is implemented. We begin with an initial-stage approximation to estimate the starting values, followed by delay approximation using Gauss's forward interpolation, and solve the remaining points using the APC scheme.

Algorithm 1: Algorithm of APC Scheme

Input: Step size h, initial value u_0, delay ratio q, final time b
Output: Numerical solution u_k for $t_k \in [0, b]$

/* Initialization */
Set $t_0 = 0$, compute $N = \frac{b}{h}$;
Set $u_0 = u(t_0)$;
for $i = 0$ *to* 2 **do**
    $u_{i+1} = u_i + h \cdot f(t_i, u_i, \texttt{ApproxDelay}(qt_i))$;

/* APC Iteration */
for $k = 3$ *to* $N - 1$ **do**
    $f_k = f(t_k, u_k, \texttt{ApproxDelay}(qt_k))$;
    $f_{k-1} = f(t_{k-1}, u_{k-1}, \texttt{ApproxDelay}(qt_{k-1}))$;
    $f_{k-2} = f(t_{k-2}, u_{k-2}, \texttt{ApproxDelay}(qt_{k-2}))$;
    /* Predictor */
    $u_{k+1}^{(p)} = u_{k-3} + \frac{4h}{3}(2f_k - f_{k-1} + 2f_{k-2})$;
    /* Corrector */
    $u_{k+1}^{c} = u_{k-1} + \frac{h}{3}(f_{k+1}^{(p)} + 4f_k + f_{k-1})$;
    $f_{k+1}^{c} = f(t_{k+1}, u_{k+1}^{c}, \texttt{ApproxDelay}(qt_{k+1}))$;

Subroutine: $\texttt{ApproxDelay}(qt)$
if $qt = t_j$ **then**
    return u_j;
else if $qt \in [t_j, t_{j+1}]$ **then**
    Approximate $u(qt)$ using Gauss's forward interpolation:
    $u(qt) \approx u_0 + q\Delta u_0 + \frac{q(q-1)}{2!}\Delta^2 u_{-1} + \frac{(q+1)q(q-1)}{3!}\Delta^3 u_{-1} + \cdots$

5. Numerical Results

In this section, four tested single-pantograph time delays were taken for consideration in order to verify the feasibility and effectiveness of the adaptive predictor-corrector (APC) scheme. The step size, $h = 10^{-3}$, with the initial condition, u_0 given in each problem are used in the solution. The initial values u_1, u_2, and u_3 are determined using a preliminary numerical approach, followed by the application of the APC scheme from u_4 onwards throughout the interval.

For each problem, the results are presented in terms of the relative error (RE) at selected time points over the interval $[0, 1]$. These are compared with the fourth-order Adams-Bashforth-Moulton (ABM4) method to assess the performance. The results in Tables 1 - 4 and Figures 2 - 5 clearly demonstrate the improvement in accuracy achieved by the APC scheme, particularly in scenarios involving vanishing or small lags in the delay terms.

In the tables, the following terminologies are used: t represents time, Exact refers to the exact solution value, RE(APC) indicates the relative error of the adaptive predictor-corrector, and RE(ABM4) denotes the relative error of the fourth-order Adams-Bashforth-Moulton method. Additionally, the notation 2.3671E-10 stands for 2.3671 x 10^{-10}.

Problem 1 Rihan (2024)

$$u'(t) = \frac{1}{2}\exp\left(\frac{1}{2}t\right)u\left(\frac{1}{2}t\right) + \frac{1}{2}u(t), \qquad 0 \le t \le 1,$$
$$u(0) = 1.$$

Exact solution: $u(t) = exp(t)$.

Table 1: Numerical results of Problem 1

t	Exact	RE(APC)	RE(ABM4)
0.1	1.105170918	2.3671E-10	5.4866E-09
0.2	1.221402758	4.7742E-10	1.1501E-08
0.3	1.349858808	7.1236E-10	1.7388E-08
0.4	1.491824698	9.4173E-10	2.3126E-08
0.5	1.648721271	1.1657E-09	2.8730E-08
0.6	1.822118800	1.3846E-09	3.4203E-08
0.7	2.013752707	1.5984E-09	3.9551E-08
0.8	2.225540928	1.8074E-09	4.4779E-08
0.9	2.459603111	2.0117E-09	4.9890E-08
1.0	2.718281828	2.2116E-09	6.1757E-08

Problem 2 Muroya *et al.* (2003)

$$u'(t) = -u(t) + \frac{1}{4}u\left(\frac{1}{2}t\right) - \frac{1}{4}\exp\left(-\frac{1}{2}t\right), \qquad 0 \le t \le 1,$$
$$u(0) = 1.$$

Exact solution: $u(t) = exp(-t)$.

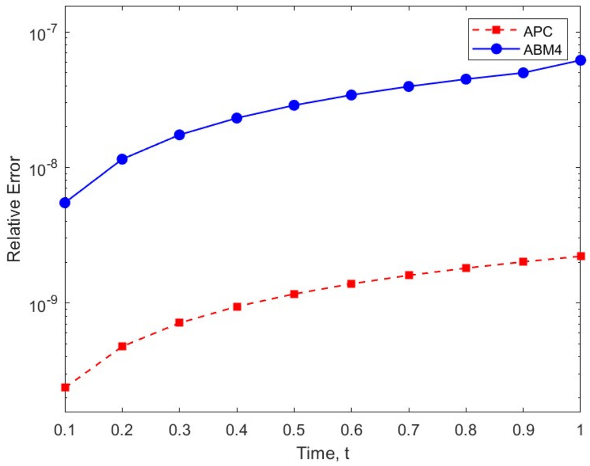

Figure 2: Relative error of APC and ABM4 methods for Problem 1

Table 2: Numerical results of Problem 2

t	Exact	RE(APC)	RE(ABM4)
0.1	0.904837418	1.3888E-10	4.0263E-09
0.2	0.818730753	2.7607E-10	7.4813E-09
0.3	0.740818221	4.2214E-10	1.1161E-08
0.4	0.670320046	5.7771E-10	1.5081E-08
0.5	0.606530660	7.4342E-10	1.9257E-08
0.6	0.548811636	9.1999E-10	2.3708E-08
0.7	0.496585304	1.1082E-09	2.8453E-08
0.8	0.449328964	1.3088E-09	3.3513E-08
0.9	0.406569660	1.5226E-09	3.8910E-08
1.0	0.367879441	1.7507E-09	4.4667E-08

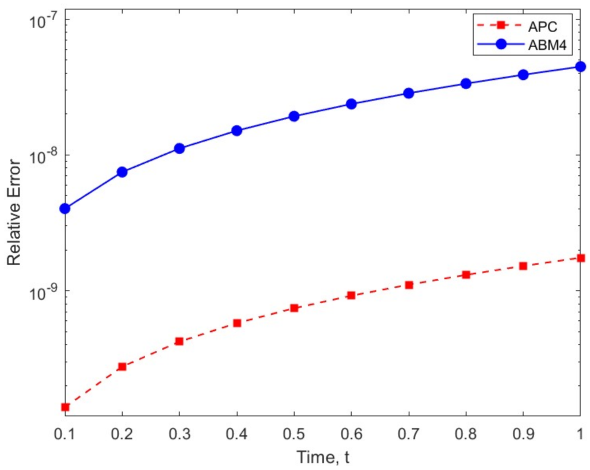

Figure 3: Relative error of APC and ABM4 methods for Problem 2

Problem 3 Ghomanjani & Shateyi (2020)

$$u'(t) = -\frac{5}{4}\exp\left(-\frac{1}{4}t\right)u\left(\frac{4}{5}t\right), \qquad 0 \le t \le 1,$$
$$u(0) = 1.$$

Exact solution: $u(t) = exp(-1.25t)$.

Table 3: Numerical results of Problem 3

t	Exact	RE(APC)	RE(ABM4)
0.1	0.882496903	1.2487E-10	1.5679E-08
0.2	0.778800783	2.5457E-10	5.3525E-08
0.3	0.687289279	3.8771E-10	1.3992E-07
0.4	0.606530660	5.2444E-10	3.2408E-07
0.5	0.535261429	6.6495E-10	5.2711E-07
0.6	0.472366553	8.0941E-10	9.2147E-07
0.7	0.416862020	9.5803E-10	2.6846E-06
0.8	0.367879441	1.1110E-09	3.5629E-06
0.9	0.324652467	1.2686E-09	4.6463E-06
1.0	0.286504797	1.4311E-09	6.4812E-06

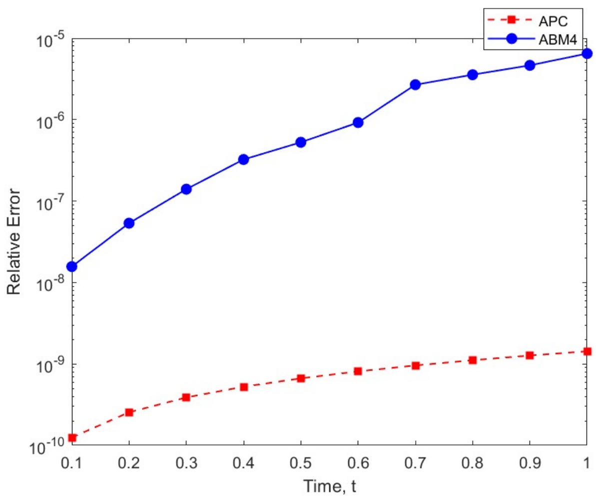

Figure 4: Relative error of APC and ABM4 methods for Problem 3

Problem 4 Doha *et al.* (2014)

$$u'(t) + u(t) - \frac{1}{10}u\left(\frac{1}{5}t\right) = -\frac{1}{10}\exp\left(-\frac{1}{5}t\right), \qquad 0 \le t \le 1,$$
$$u(0) = 1.$$

Exact solution: $u(t) = exp(-t)$.

Table 4: Numerical results of Problem 4

t	Exact	RE(APC)	RE(ABM4)
0.1	0.904837418	3.0598E-10	1.8485E-09
0.2	0.818730753	5.2907E-10	4.0845E-09
0.3	0.740818221	7.7094E-10	7.5210E-09
0.4	0.670320046	1.0332E-09	2.5707E-08
0.5	0.606530660	1.3175E-09	5.6460E-08
0.6	0.548811636	1.6257E-09	1.9639E-07
0.7	0.496585304	1.9598E-09	3.9280E-07
0.8	0.449328964	2.3219E-09	7.6261E-07
0.9	0.406569660	2.7145E-09	8.7072E-07
1.0	0.367879441	3.1399E-09	9.3679E-07

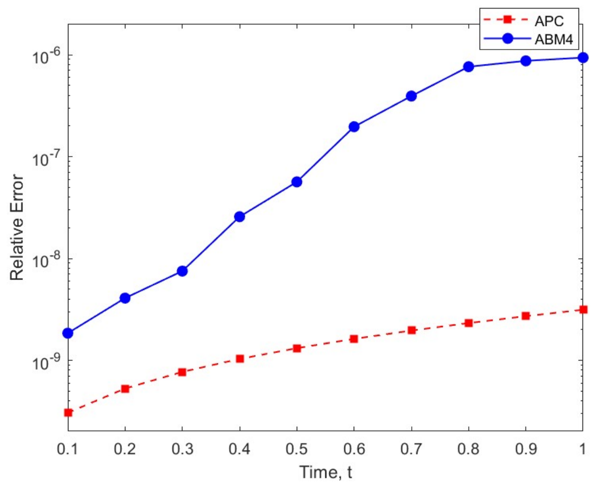

Figure 5: Relative error of APC and ABM4 methods for Problem 4

6. Discussion

Four tested problems of single-pantograph time delay were numerically solved by using adaptive predictor-corrector scheme with constant step size. Gauss's forward interpolation algorithm is utilized to assess the delay term found in the equation. This is because this type of interpolation is trivial to implement in the delay part of equations and seems to achieve the required precision based on the results obtained.

Tables 1 - 4 display a comparison between the exact solution as well as the relative error at each point, t, for solving single-pantograph time delay differential equations at constant step size, $h = 10^{-3}$ for Problems 1 - 4 respectively. From the obtained results in Table 1 - 4, we can see that the adaptive predictor-corrector method obtained superior maximum value for relative errors which is the estimated significant digits is up to 10^{-10} for all problems.

Figures 2 - 5 further illustrate the comparative performance of both methods across all test problems. The APC scheme exhibits a steeper decline in RE over time, highlighting its faster convergence rate. In contrast, the ABM4 method shows a more gradual or even stagnant error reduction, particularly in problems where delay terms fall within intermediate mesh intervals. This demonstrates that the APC method not only achieves higher precision but also maintains numerical stability in challenging delay configurations.

7. Conclusions

In this study, we have discussed the application of adaptive predictor-corrector scheme with constant step size for solving a class of single-pantograph time delay differential equations. Note that, the strategy of approximating the pantograph delay using Gauss's forward interpolation formula provides a good approximation value, which results in an acceptable small value of error when compared to an exact solution.

The convergence analysis of the proposed method is also presented which shows that the method is converge to the exact solution. Thus, we can conclude that the Gauss's forward interpolation formula combined with adaptive predictor-corrector scheme can effectively handle the single-pantograph time delay problem. The current analysis's capacity to solve a single pantograph-type time delay might merit more in-depth future research on other classes of multiple delay pantograph equations.

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Department of Mathematical Sciences
Faculty of Intelligent Computing
Universiti Malaysia Perlis (UniMAP)
Kampus Alam UniMAP Pauh Putra
02600 Arau
Perlis, MALAYSIA
*E-mail: nuraishahjamaludin@studentmail.unimap.edu.my, hudaaziz@unimap.edu.my**

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*Corresponding author