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ON b-COLORING OF UNICYCLIC AND BICYCLIC GRAPHS

(*Suatu b-Pewarnaan bagi Graf Unikitaran dan Dwikitaran*)

RIDHO ALFARISI, SHARIFAH KARTINI SAID HUSAIN*, ARIKA INDAH KRISTIANA & WITRIANY BASRI

ABSTRACT

The present work focuses on establishing the **b**-coloring characteristics of both unicyclic and bicyclic graphs. Recall that a b-coloring of a graph G using k distinct colors is a valid k-coloring where every color partition contains at least one vertex adjacent to vertices of all $k-1$ other color classes. The **b**-chromatic number of G, symbolized as $\varphi(G)$, represents the highest integer k for which such a b-coloring of G is feasible. In this paper, we determine the b-coloring of unicyclic and bicyclic graphs.

Keywords: b-coloring; b-chromatic number; unicyclic; bicyclic

ABSTRAK

Suatu b-pewarnaan graf G dengan k warna ialah k-pewarnaan G yang betul supaya dalam setiap kelas warna wujud satu bucu yang mempunyai jiran dalam semua kelas warna $k-1$ yang lain. b-nombor kromatik bagi graf G, dilambangkan dengan $\varphi(G)$, ialah maksimum k yang G mempunyai b-pewarnaan sebanyak k warna. Dalam kertas ini, b-pewarnaan bagi graf unikitaran dan dwi kitaran ditentukan.

Kata kunci: b-pewarnaan, b-nombor kromatik b, unikitaran dan bikitaran

1. Introduction

We consider that all graphs in this paper are finite, simple, and connected, for detailed definitions of graph, see Gross *et al.* (2014) and Chartrand and Lesniak (2000). There are some topics in coloring problems, include vertex coloring, edge coloring, r-dynamic, irregular coloring, total coloring, star coloring, and equitable coloring.

Applications of coloring problems, namely compilation in the work schedule (Silitonga 2023), setting of traffic lights, and electric circuit problems (Balqis *et al.* 2022). There are some applications of b-coloring include automatic document recognition (Gaceb *et al.* 2009), multifaceted coding of messages (Medini *et al.* 2024)), and a diversity algorithm of nested rollout policy adaptation (Yang *et al.* 2023). The b-coloring ensures that each cluster (color class) has at least one representative (b-vertex) which is adjacent to vertices in all other clusters, facilitating effective inter-cluster communication and analysis. This approach has been applied successfully to benchmark datasets and real medical data, demonstrating its utility in organizing complex data structures for tasks like patient classification and treatment optimization (Elghazel *et al.* 2008).

A b-coloring of a graph G with k distinct colors is defined as a proper k-coloring where, for every color assigned, at least one vertex belonging to that color class is adjacent to vertices from all the remaining $k-1$ color classes. The b-chromatic number of a graph G, symbolized as $\varphi(G)$, was first conceived by Irving and Manlove (1999) and represents the largest possible integer k for which such a b-coloring of G can be achieved. It's a fundamental observation that any proper coloring of G using $\chi(G)$ colors inherently constitutes a b-coloring, with $\chi(G)$ representing the chromatic number of the graph G.

Proposition 1. *Bounds of b-coloring of graphs G with maximum degree $\Delta(G)$ as follows*

$$\chi(G) \leq \varphi(G) \leq \Delta(G) + 1.$$

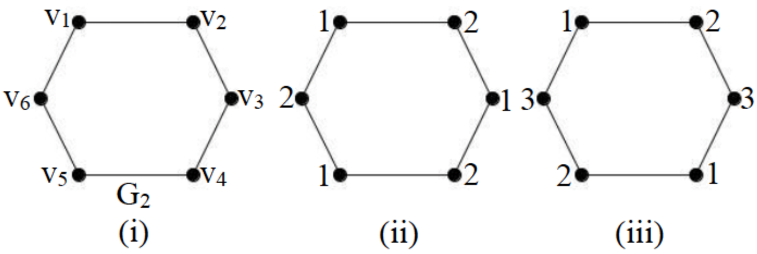


Figure 1: (i) Cycle graph C_6; (ii) proper $2-$coloring; and (iii) $b-$coloring with 3 color

We choose $G_2 = C_6$ with $\Delta(G_2) = 2$. Figure 1 (ii) shows a proper 2-coloring as well as a b-coloring of G_2 with $\chi(G_2) = 2$. While Figure 1 (iii) shows a true 3-coloring as well as a b-coloring of G_2. Since there is no b-coloring of G_2 using 4 colors, then $\varphi(G_2) = 3$.

There are some previous results of b-coloring in some families graph, namely the regular graph (Blidia *et al.* 2009), the windmill graph (Venkatachalam & Vivin 2013), bipartite graph (Blidia *et al.* 2012) and the graph resulting operation namely the cartesian product (Javadi & Omoomi 2012), the product of the path and cycle (Afrose & Fathima 2020), the corona and shadow in C_n^k (Kumar 2016), and corona product (Vernold & Venkatachalam 2012).

This paper deals with the problem of b coloring of unicyclic and bicyclic graphs. The unicyclic graph is a graph that has only one cycle, while the bicyclic graph is a graph that has only two cycles. An example of a unicyclic graph and a bicyclic graph can be seen in Figure 2.

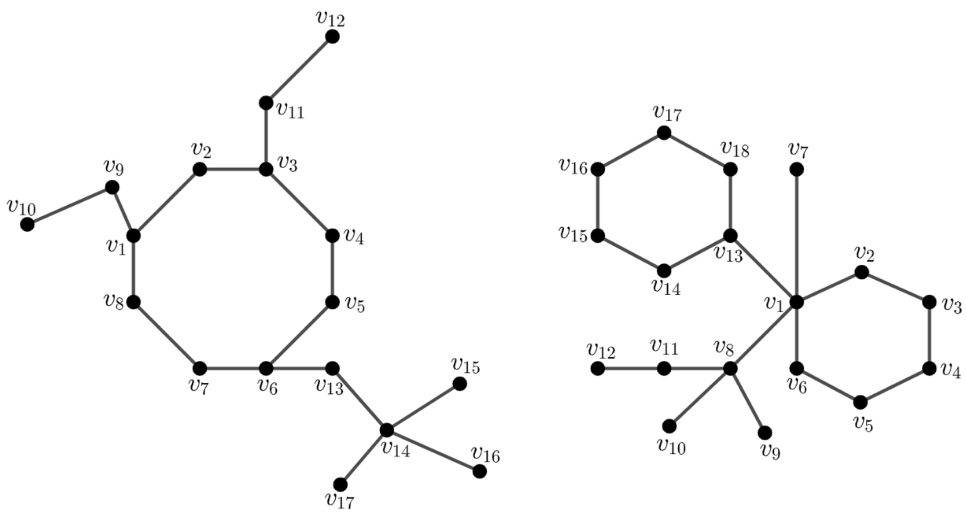


Figure 2: Unicyclic (left) and bicyclic (right) graphs

2. Results and Discussion

This paper investigates the b-chromatic number of unicyclic and bicyclic graphs. Tadpole graph $Tp_{n,m}$ and corona product of cycle and nullgraph $C_n \odot mK_1$ represent as unicyclic graphs while Barbell graph $B_{n,m}$ as bicyclic graphs.

Theorem 1 presents the b-coloring of unicycle graph namely Tadpole graph, while Theorem 2 shows the b-coloring of the bicyclic graph namely Barbell graph.

Theorem 1. *Let $Tp_{n,m}$ be a tadpole graph with a cycle C_n for $n \geq 5$ and $m \geq 4$, then $\varphi(Tp_{n,m}) = 4$.*

Proof. Tadpole $Tp_{n,m}$ has a cycle C_n. Let the vertex and edge sets of $Tp_{n,m}$ as

$$V(Tp_{n,m}) = \{u_i | 1 \leq i \leq n\} \cup \{v_j | 1 \leq j \leq m\},$$

$$E(Tp_{n,m}) = \{u_i u_{i+1} | 1 \leq i \leq n-1, u_1 u_n\} \cup \{u_1 v_1, v_j v_{j+1} | 1 \leq j \leq m-1\}.$$

Since the maximum degree of $Tp_{n,m}$ is 3 then Proposition 1 shows $\varphi(Tp_{n,m}) \leq 4$.

Now consider the following 4-coloring $(1, 2, 3, 4)$ of $Tp_{n,m}$ that have the patterns of assign color in vertices of $Tp_{n,m}$ as follows

For $n \equiv 0, 2(mod\, 3)$,

$$f(u_i) = \begin{cases} 1, & \text{if } i \equiv 1(mod\, 3) \\ 2, & \text{if } i \equiv 2(mod\, 3) \\ 3, & \text{if } i \equiv 0(mod\, 3) \end{cases}$$

$$f(v_j) = \begin{cases} 2, & \text{if } j \equiv 2(mod\, 3) \\ 3, & \text{if } j \equiv 1(mod\, 3) \\ 4, & \text{if } j \equiv 0(mod\, 3) \end{cases}$$

For $n \equiv 1(mod\, 3)$,

$$f(u_i) = \begin{cases} 1, & \text{if } i \equiv 1(mod\, 3), i \neq n \\ 2, & \text{if } i \equiv 2(mod\, 3), i = n \\ 3, & \text{if } i \equiv 0(mod\, 3) \end{cases}$$

$$f(v_j) = \begin{cases} 2, & \text{if } j \equiv 2(mod\, 3) \\ 3, & \text{if } j \equiv 1(mod\, 3) \\ 4, & \text{if } j \equiv 0(mod\, 3), j = 1 \end{cases}$$

Based on the assigned 4 colors in $Tp_{n,m}$, the color classes are as follows.

$$\begin{aligned} C_1 &= \{u_i; i \equiv 1(mod\, 3), n \equiv 0, 2(mod\, 3)\} \cup \{u_i; i \equiv 1(mod\, 3), i \neq n, n \equiv 1(mod\, 3)\}, \\ C_2 &= \{u_i; i \equiv 2(mod\, 3), n \equiv 0, 2(mod\, 3)\} \cup \{u_i; i \equiv 2(mod\, 3), i = n, n \equiv 1(mod\, 3)\} \\ &\quad \cup \{v_j; j \equiv 2(mod\, 3)\}, \\ C_3 &= \{u_i; i \equiv 0(mod\, 3), n \equiv 0, 1, 2(mod\, 3)\} \cup \{v_j; j \equiv 1(mod\, 3), j \neq 1\}, \\ C_4 &= \{v_j; j \equiv 0(mod\, 3), j = 1\}. \end{aligned}$$

It is known that each color class contains at least one vertex in color class adjacent to other color classes. Thus, $\varphi(Tp_{n,m}) \geq 4$. Therefore, we conclude that $\varphi(Tp_{n,m}) = 4$. □

Theorem 2. *Let $B_{n,m}$ be a Barbell graph with two cycles C_n and C_m, then $\varphi(B_{n,m}) = 4$.*

Proof. Barbell $B_{n,m}$ has two cycles C_n and C_m which two cycles are connected by a path P_s. The path P_s is called bridge. Let

$$V(B_{n,m}) = \{u_i | 1 \leq i \leq n\} \cup \{w_j | 1 \leq j \leq s\} \cup \{v_k | 1 \leq k \leq m\},$$

and

$$E(B_{n,m}) = \{u_i u_{i+1} | 1 \leq i \leq n-1, u_1 u_n\} \cup \{u_1 w_1, w_j w_{j+1} | 1 \leq j \leq m-1\}$$
$$\cup \{w_s v_1, v_k u_{k+1} | 1 \leq k \leq m-1, v_1 v_m\}$$

be vertex and edge sets of Barbell graph, respectively. Proposition 1 shows that $\varphi(B_{n,m}) \leq 4$, since the maximum degree of $B_{n,m}$ is 3.

Consider the following 4-coloring $(1, 2, 3, 4)$ of $B_{n,m}$. The patterns of assign color in vertices of $B_{n,m}$ are as follows

- Assign color in cycle C_n,

$$f(u_i) = \begin{cases} 1, & \text{if } i \equiv 1(mod\, 3), n \equiv 0, 2(mod\, 3) \\ 2, & \text{if } i \equiv 2(mod\, 3), n \equiv 0, 2(mod\, 3) \\ 3, & \text{if } i \equiv 0(mod\, 3), n \equiv 0, 2(mod\, 3) \end{cases}$$

$$f(u_i) = \begin{cases} 1, & \text{if } i \equiv 1(mod\, 3), i \neq n, n \equiv 1(mod\, 3) \\ 2, & \text{if } i \equiv 2(mod\, 3), i = n, n \equiv 1(mod\, 3) \\ 3, & \text{if } i \equiv 0(mod\, 3), n \equiv 1(mod\, 3) \end{cases}$$

- Assign color in bridge between C_n and C_m

$$f(w_j) = \begin{cases} 2, & \text{if } j \equiv 2(mod\, 3) \\ 3, & \text{if } j \equiv 1(mod\, 3) \\ 4, & \text{if } j \equiv 0(mod\, 3), j = 1 \end{cases}$$

- Assign color in cycle C_m.
  For $s \equiv 1(mod\, 3)$,

$$f(v_k) = \begin{cases} 2, & \text{if } k \equiv 1(mod\, 3), k \neq m, m \equiv 1(mod\, 3) \\ 3, & \text{if } k \equiv 0(mod\, 3), m \equiv 1(mod\, 3) \\ 4, & \text{if } k \equiv 2(mod\, 3), k = m, m \equiv 1(mod\, 3) \end{cases}$$

$$f(v_k) = \begin{cases} 2, & \text{if } k \equiv 1(mod\, 3), m \equiv 0, 2(mod\, 3) \\ 3, & \text{if } k \equiv 0(mod\, 3), m \equiv 0, 2(mod\, 3) \\ 4, & \text{if } k \equiv 2(mod\, 3), m \equiv 0, 2(mod\, 3) \end{cases}$$

  For $s \equiv 2(mod\, 3)$,

$$f(v_k) = \begin{cases} 2, & \text{if } k \equiv 2(mod\, 3), k \neq m, m \equiv 1, 2(mod\, 3) \\ 3, & \text{if } k \equiv 1(mod\, 3), k \neq 1, m \equiv 1, 2(mod\, 3) \\ 4, & \text{if } k \equiv 0(mod\, 3), k = 1, m \equiv 1, 2(mod\, 3) \end{cases}$$

$$f(v_k) = \begin{cases} 2, & \text{if } k \equiv 2(mod\, 3), m \equiv 0(mod\, 3) \\ 3, & \text{if } k \equiv 1(mod\, 3), k \neq 1, k = m, m \equiv 0(mod\, 3) \\ 4, & \text{if } k \equiv 0(mod\, 3), k \neq m, m \equiv 0(mod\, 3) \end{cases}$$

  For $s \equiv 0(mod\, 3)$,

$$f(v_k) = \begin{cases} 2, & \text{if } k \equiv 2(mod\, 3), m \equiv 0, 2(mod\, 3) \\ 3, & \text{if } k \equiv 1(mod\, 3), m \equiv 0, 2(mod\, 3) \\ 4, & \text{if } k \equiv 0(mod\, 3), m \equiv 0, 2(mod\, 3) \end{cases}$$

$$f(v_k) = \begin{cases} 2, & \text{if } k \equiv 2(mod\,3), k = m, m \equiv 1(mod\,3) \\ 3, & \text{if } k \equiv 1(mod\,3), k \neq m, m \equiv 1(mod\,3) \\ 4, & \text{if } k \equiv 0(mod\,3), m \equiv 1(mod\,3) \end{cases}$$

For $n \equiv 1(mod\,3)$,

$$f(u_i) = \begin{cases} 1, & \text{if } i \equiv 1(mod\,3), i \neq n \\ 2, & \text{if } i \equiv 2(mod\,3), i = n \\ 3, & \text{if } i \equiv 0(mod\,3) \end{cases}$$

$$f(v_j) = \begin{cases} 2, & \text{if } j \equiv 2(mod\,3) \\ 3, & \text{if } j \equiv 1(mod\,3) \\ 4, & \text{if } j \equiv 0(mod\,3), j = 1 \end{cases}$$

Based on the assign 4 colors in $B_{n,m}$, we have four color classes which in each color classes, at least one vertices in color class adjacent to other color classes. Thus $\varphi(B_{n,m}) \geq 4$. Hence, we obtain $\varphi(B_{n,m}) = 4$. □

Figure 3 below shows an example of a Barbel graph.

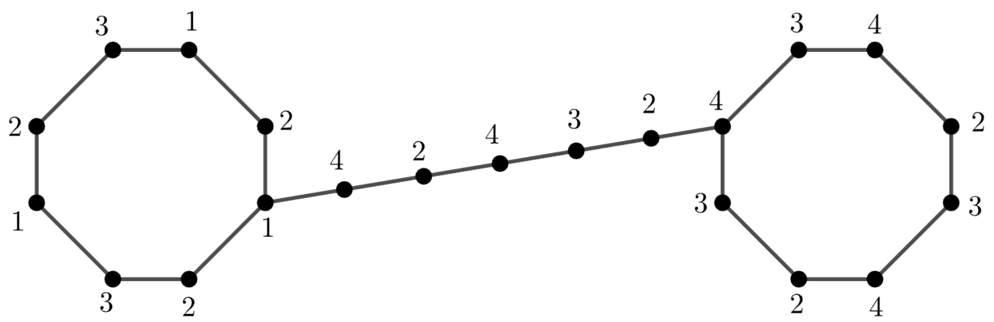


Figure 3: Barbell graph $B_{8,8}$

Next, the b-coloring of unicycle graph namely corona product of cycle C_n and null graph mK_1, denoted by $C_n \odot mK_1$, is determined. Let

$$V(C_n \odot mK_1) = \{u_i; 1 \leq i \leq n\} \cup \{u_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m\}$$

be a vertex set of $C_n \odot mK_1$ and

$$E(C_n \odot mK_1) = \{u_1u_n, u_iu_{i+1}; 1 \leq i \leq n-1\} \cup \{u_iu_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m\}$$

be a edge set of $C_n \odot mK_1$. The maximum degree of $C_n \odot mK_1$ is $m + 2$.

Theorem 3. *Let $G = C_n \odot mK_1$ be a unicyclic graph with a subgraph cycle C_n, then*

$$\varphi(G) = \begin{cases} \Delta(G) - 1, & \text{if } m = n, \\ \Delta(G), & \text{if } m = n - 1, \\ \Delta(G) + 1, & \text{if } m \leq n - 2, \\ n + 1, & \text{if } m > n. \end{cases}$$

Proof. Consider four cases as follows:
Case 1. For $m = n$.

Consider the following $(\Delta(G)-1)$-coloring $(1,2,3,\ldots,\Delta(G)-1)$ of G. We have the pattern of assign color in vertices of G as follows.

$$f(u_i) = i, 1 \leq i \leq n,$$

$$f(u_{i,j}) = i + j; 1 \leq i \leq n, 1 \leq j \leq m - i + 1,$$

$$f(u_{i,j}) = i + j - m - 1; 2 \leq i \leq n, m - i + 2 \leq j \leq m.$$

Based on the assign $(\Delta(G) - 1)$ colors in G. We know that u_i adjacent to $u_{i,j}$. For every vertices u_i have different color which we have n color classes. The vertices of color class C_{n+1} is pendant vertices. The pendant vertices $u_{i,j}$ adjacent to u_i so that each color class which there are at least one vertices in color class adjacent to other color classes. Therefore we obtain the bounds of $\varphi(G) \geq \Delta(G) - 1$.

Furthermore, using the Proposition 1, we know that $\varphi(G) \leq \Delta(G) + 1$. Hence, we get $\varphi(G) = \Delta(G) - 1$. We try to give $\Delta(G)$ colors in G. Consider two conditions for color vertex in G as follows

(1) Since we use $\Delta(G)$ colors in G, then every vertex $u_i \in V(G)$ given $\Delta(G) - 2$ colors so $f(u_i) = i$ with $1 \leq i \leq \Delta(G) - 2$.
(2) The colors $\Delta(G) - 1$ and $\Delta(G)$ delivery to pendant vertices. We know that the colors of vertex neighbourhood of vertex u_k are $f(u_{k-1}) = k - 1$, $f(u_{k+1}) = k + 1$, and the colors of pendant vertex are $\{1, 2, 3, \ldots, \Delta(G)\}\backslash\{k, k-1, k+1\}$ with $1 \leq k \leq \Delta(G)$. The vertices with $\Delta(G)$ and $\Delta(G) - 1$ are not adjacent, then it is contradiction.

Therefore, we get $\varphi(G) \leq \Delta(G) - 1$. Thus, $\varphi(G) = \Delta(G) - 1$.

Case 2. For $m = n - 1$.
In this case, we have $(\Delta(G))$-coloring $(1, 2, 3, \ldots, \Delta(G))$ of G and the pattern of assign color in vertices of G is

$$f(u_i) = i, 1 \leq i \leq n,$$

$$f(u_{i,j}) = j + i + 1; 1 \leq i \leq 2, 1 \leq j \leq m - 2,$$

$$f(u_{n,j}) = j + 1; 1 \leq j \leq m - 2,$$

$$f(u_{i,j}) = n + 1; i \in \{1, 2, n\}, m - 1 \leq j \leq m,$$

$$f(u_{i,j}) = j; 3 \leq i \leq n, 1 \leq j \leq i - 2,$$

$$f(u_{i,j}) = j + 3; 3 \leq i \leq n, i - 1 \leq j \leq m - 2,$$

$$f(u_{i,j}) = n + 1; 3 \leq i \leq n, m - 1 \leq j \leq m.$$

Based on the assign $(\Delta(G))$ colors in G. We know that u_i adjacent to $u_{i,j}$. Every vertices u_i have different colors which we have n color classes. The vertices of color class C_{n+1} is pendant vertices. The pendant vertices $u_{i,j}$ adjacent to u_i so that each color classes which there are at least one vertices in color classes adjacent to other color classes. It shows that the bounds of $\varphi(G) \geq \Delta(G)$.

Next, by using the Proposition 1, we get $\varphi(G) \leq \Delta(G) + 1$. Hence, $\varphi(G) = \Delta(G)$. We try to give $\Delta(G) + 1$ colors in G. There are two conditions for color vertex in G as follows

(1) Since we use $\Delta(G) + 1$-colors in G, then every vertex $u_i \in V(G)$ given $\Delta(G) - 1$ color so $f(u_i) = i$ with $1 \leq i \leq \Delta(G) - 1$.
(2) The colors $\Delta(C_n \odot mK_1)$ and $\Delta(G)$ delivery to pendant vertices. We know that the colors of vertex neighbourhood of vertex u_k are $f(u_{k-1}) = k - 1$, $f(u_{k+1}) = k + 1$, while the colors of pendant vertex are $\{1, 2, 3, \ldots, \Delta(G \odot H)\}\backslash\{k, k-1, k+1\}$ with

$1 \leq k \leq \Delta(G \odot H) + 1$. The vertices with $\Delta(G \odot H)$ and $\Delta(G \odot H) + 1$ are not adjacent, then it is contradiction.

Based on the above cases, its show that $\varphi(G) \leq \Delta(G)$. Thus, $\varphi(G) = \Delta(G)$.

Case 3. For $m \leq n - 2$.
We show the bounds of $\varphi(G) \geq m + 3$. Consider the following $(m + 3)$-coloring. We have the pattern of assign color in vertices of G as follows.

(1) We give color in every vertex $u_i, 1 \leq i \leq n$ with periodic pattern as $\{u_1, u_2, \ldots, u_n\} = \{1, 2, 3, \ldots, m+3, u_{m+4}, \ldots, u_n\}$ which color in vertices $\{u_{m+4}, \ldots, u_n\}$ give colors $\{1, 2, \ldots, m + 3\}$.
(2) Every pendant vertex $u_{i,j}$ give color with pattern as for $f(u_{k,j}) \neq f(u_{k-1}) \neq f(u_{k+1})$ such that let $f(u_k) = a$, then color in $u_{k,j}$ with $1 \leq j \leq m$ does not contains colors $\{a, f(u_{k-1}), f(u_{k+1})\}$ and color $f(u_{k,r}) \neq f(u_{k,s})$ with $1 \leq r \neq s \leq m$. Thus, color in vertices $\{u_k, u_{k-1}, u_{k+1}, u_{k,j}\}$ have different colors with pattern $\{1, 2, 3, 4, \ldots, m + 3\}$.

Based on the assign $(m + 3)$ colors in G. Since u_i adjacent to $u_{i,j}$ and every vertices u_i have different colors, we have $m + 3$ color classes. The pendant vertices $u_{i,j}$ adjacent to u_i so that each color classes which there are at least one vertices in color class adjacent to other color classes. Therefore, Proposition 1 gives $\varphi(G) \leq \Delta(G) + 1 = m + 3$. Thus, $\varphi(G) = \Delta(G) + 1$.

Case 4. For $m > n$.
Let $(1, 2, 3, \ldots, n + 1)$ be $(n + 1)$-coloring of G and the pattern of assign $(n + 1)$ colors in vertices of G as follows

$$f(u_i) = i, 1 \leq i \leq n,$$

$$f(u_{i,j}) = i + j; 1 \leq i \leq n, 1 \leq j \leq n - i + 1,$$
$$f(u_{i,j}) = n + 1; 1 \leq i \leq n, n - i + 2 \leq j \leq m - i + 1,$$
$$f(u_{i,j}) = i + j - m - 1; 1 \leq i \leq n, m - i + 2 \leq j \leq m.$$

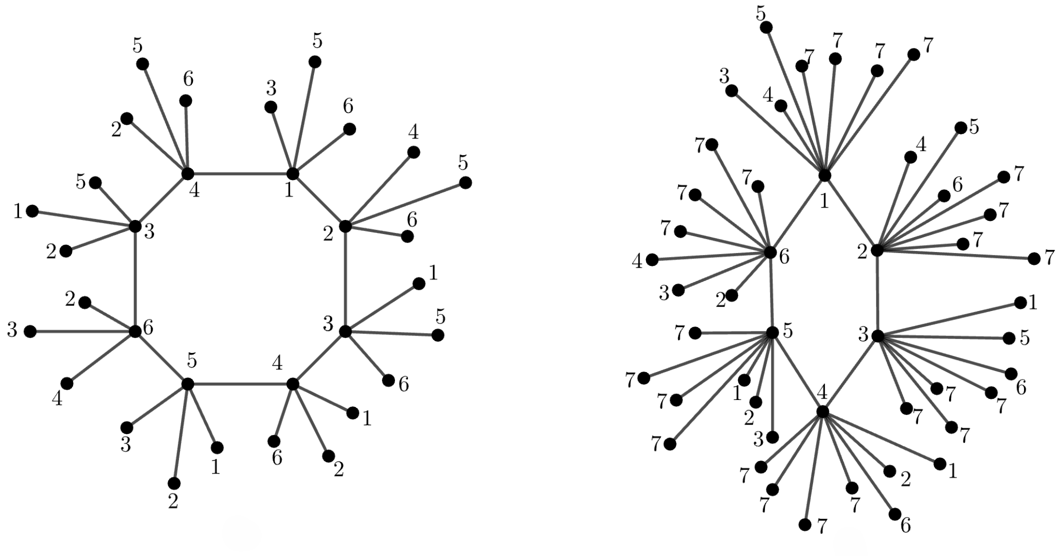
Since u_i adjacent to $u_{i,j}$. For every vertices u_i have different color which we have n-color classes. The vertices of color class C_{n+1} is pendant vertices. The pendant vertices $u_{i,j}$ adjacent to u_i so that each color class which there are at least one vertices in color class adjacent to other color classes. We show the bounds of $\varphi(G) \geq n + 1$.

From the Proposition 1, we have $\varphi(G) \leq \Delta(G) + 1 = m + 3$. Hence, $\varphi(G) = n + 1$.Now, we try to give $n + 2$ colors in G. The following are some conditions for color vertex in G

(1) Since we use $n + 1$ colors in G, then every vertex $u_i \in V(G)$ given n colors so $f(u_i) = i$ with $1 \leq i \leq n$.
(2) The colors $n + 1$ and $n + 2$ delivery to pendant vertices. The colors of vertex neighbourhood of vertex u_k are $f(u_{k-1}) = k - 1$, $f(u_{k+1}) = k + 1$, and the colors of pendant vertices are $\{1, 2, 3, \ldots, n + 2\} \backslash \{k, k - 1, k + 1\}$ with $1 \leq k \leq n + 2$. The vertices with colors $n + 1$ and $n + 2$ are not adjacent, then it is contradiction.

From the above conditions, it is clear that $\varphi(G) \leq n + 1$. Thus, $\varphi(G) = n + 1$. □

The following figures (Figure 4 (a) and (b)) show the corona product of $C_n \odot mK_1$ for the cases $m < n$ and $m > n$.



(a) (b)

Figure 4: (a) $m < n$ and (b) $m > n$

3. Conclusion

This study successfully ascertains the precise value of the b-chromatic number of for unicyclic and bicyclic graphs. Based on the results, there is exists a graph that attains the upper bounds. An open problem in this topic is as follows:

Open Problem 1. *Investigate the bounds of b-chromatic number of unicyclic and bicyclic.*

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