Journal of Quality Measurement and Analysis *JQMA* 21(4) 2025, 41-53
e-ISSN: 2600-8602 *http://www.ukm.my/jqma*
https://doi.org/10.17576/jqma.2104.2025.03

A NEW ATTRIBUTE REDUCTION ALGORITHM UNDER TOLERANCE-BASED RELATION FOR ROUGH NEUTROSOPHIC DECISION SYSTEM

(Algoritma Pengurangan Atribut Baru Berdasarkan Toleransi untuk Sistem Neutrosofik Kasar)

SITI NUR AISYAH MOHD ZAINAL, AHMAD TERMIMI AB GHANI*,
MOHD LAZIM ABDULLAH & NING NING PENG

ABSTRACT

Rough set has been successfully combined with other mathematical frameworks to improve attribute reduction. In particular, attribute reduction is essential for processing and analyzing the Rough Neutrosophic Decision System. The rough neutrosophic set provides an effective framework for managing vagueness, inconsistency and incomplete information. This hybrid model allows a more flexible representation of real-world data by incorporating truth, indeterminacy and falsity membership functions. This study introduces a novel attribute reduction technique based on tolerance relations in the context of rough neutrosophic sets, employing rough neutrosophic numbers to express information values. The proposed method includes the formation of lower and upper approximations and defines the degree of dependency between decision attributes and conditional attributes. An algorithm is developed to implement the approach and a detailed example in coastal erosion is provided to validate its practical application. Experimental outcomes demonstrate that the method efficiently identifies relevant and non-relevant attributes, thereby enhancing the decision-making process. The proposed method not only improves the precision of data but also strengthens the robustness of intelligent systems when dealing with complex and uncertain datasets.

Keywords: tolerance relation; attribute reduction; rough neutrosophic set; similarity relation

ABSTRAK

Set kasar telah berjaya bergabung dengan rangka kerja matematik lain untuk meningkatkan proses pengurangan atribut. Khususnya, pengurangan atribut memainkan peranan penting dalam pemprosesan dan analisis Sistem Maklumat Set Netrosofik Kasar.Set netrosofik kasar menyediakan kerangka kerja yang berkesan untuk menangani kekaburan, ketidakkonsisten dan maklumat yang tidak lengkap. Model hybrid ini membolehkan perwakilan data sebenar yang lebih fleksibel dengan menggabungkan fungsi keahlian kebenaran, ketidaktentuan and kepalsuan. Kajian ini memperkenalkan eknik baru bagi penggurangan atribut berdasarkan hubungan toleransi dalam konteks set netrosofik kasar mewakili nilai matlumat. Kaedah yang dicadangkan merangkumi pembentukan penghampir bawah dan atas serta mentakrifkan tahap pergantungan antara atribut Keputusan dan atribut bersyarat. Satu algoritma dibagunkan untuk melaksankan pendekatan ini dan contoh terperinci dalam isu hakisan pantai disediakan untuk mengesahkan aplikasi praktikalnya. Hasil eksperimen menunjukkan bahawa kaedah ini secara berkesan mengenal pastu atribut yang relevan dan tidak relevan, seterusnya meningkatkan proses membuat keputusan. Kaedah yang dicadangkan bukan Sahaja mempertingkatkan ketepatan data, tetapi juga mengukuhkan ketahanan system pintar dalam menghadapi set data yang kompleks dan tidak menentu.

Kata kunci: Hubungan toleransi; pengurangan atribut; neutrosofik kasar ; hubungan kesamaan

1. Introduction

The need for effective data analysis techniques has increased recently due to the exponential growth of data. Attribute reduction, known as feature selection, is a crucial preprocessing step in this context and vital for data mining and machine learning. It simplifies data by removing irrelevant or redundant features, thus improving computational efficiency and model performance. To solve this issue, a variety of attribute reduction methods have been suggested and widely implemented. Su *et al.* (2020) proposed an efficient attribute reduction using rough set theory. They utilized chi-square statistics and condition entropy to assess the importance of attributes. Their approach integrates both forward and backward selection strategies to guide the reduction process effectively. Chebrolu & Sanjeevi (2017) developed a hybrid approach integrating the artificial bee colony algorithm that integrates discretization and attribute reduction within rough set framework. Liang *et al.* (2024) later proposed the Incomplete Knowledge Attribute Reduction (IKAR) algorithm to address the limitations of traditional attribute reduction.

Rough set theory, first introduced by Pawlak (1982), offers a strong mathematical basis for identifying relevant features in decision systems, particularly those using complete information. Attribute reduction in rough sets typically involves two major techniques: indiscernibility matrix and evaluation based on attribute significance measures. Skowron & Rauszer (1992) defined the discernibility matrix and function as key tools in analyzing information systems. These concepts enable the development of algorithms for tasks such as identifying rough definability, generating reducts and cores and discovering attribute dependencies. Jensen & Shen (2004) introduced semantics-preserving dimensionality reduction to identify the most predictive features while retaining the meaning of the data, especially high-dimensional task like text and web content classification. The rough set theory's scope has been expanded to facilitate research in intelligent systems that operate under conditions of uncertain, insufficient, and incomplete information. Several techniques based on rough set theory have been developed to extract decision rules from datasets structured as decision tables (information system). At the core of Pawlak's rough set framework lies the equivalence relation, which underpins the formation of lower and upper approximations. Classical rough set models are well-suited for analyzing categorical data, practical applications often involve real-valued attributes that describe the characteristics of objects interest (Ahmad *et al.* 2006; Alfares & Duffuaa 2008; Gong *et al.* 2008; Qian *et al.* 2008). To address this limitation, discretization techniques is frequently used to convert continuous data into categorical form before applying attribute reduction techniques. However, this transformation can lead to information loss.

Fuzzy set is well-suited for representing uncertainty and imprecise data due to its inherent flexibility and tolerance for uncertainty (Zadeh 1965). It has been effectively applied across various fields, including risk evaluation (Baser *et al*. 2023), healthcare (Gou 2021) and forecasting (Pattanayak *et al*. 2021). Beyond these traditional applications, fuzzy sets have also demonstrated value in emerging areas such as deep learning (Bonanno *et al*. 2017) and topological data analysis (Vasilakakis & Iakovidis 2023). However, in practical decision-making scenarios, relying on single membership function can be limited due to complexity of real-world uncertainty. To address this limitation, Atanassov (1986) introduced intuitionistic fuzzy sets as an extension of classical fuzzy sets which include true and false membership. This framework, regarded as a subset of context-dependent fuzzy sets, effectively addresses certain limitations inherent in conventional fuzzy set theory. Neutrosophic sets, proposed by Smarandache (2006), generalize intuitionistic fuzzy sets by introducing a third independent component to represent indeterminacy addressing challenges that fuzzy logic could not adequately resolve (Zhang *et al.* 2010). In contrast to the two memberships of intuitionistic fuzzy sets, neutrosophic sets defined as truth, falsity, and indeterminacy membership.

Neutrosophic theory enhances the capacity for modelling uncertainty and has demonstrated applicability in various domains, including optimization (Han *et al.* 2020), data mining (Yuan *et al.* 2021), decision support systems (Yazdani *et al.* 2021) and medical diagnosis (Şahin & Karabacak 2020).

Attribute reduction methods have also been adapted for fuzzy and rough fuzzy systems. Dubois and Parade (1990) introduced a fuzzy rough set (FRS) and rough fuzzy set (RFS) framework, enabling attribute reduction directly on numerical data. This framework has since been the basis for developing reduction algorithms that leverage various fuzzy rough set-based measures for handling numeric attributes without discretization. Besides, the concepts RFS and FRS have been further developed by replacing crisp binary relation within the universe with a fuzzy relation (Kondo 2006; Lin 1992). Su (2020) developed novel uncertainty measures and proposed a feature selection algorithm based on fuzzy neighborhood multigranulation rough sets. Sun (2021) proposed introduces a multilabel neighborhood rough sets-based uncertainty measures which are maximum relevance-minimum redundancy (mRMR), and heuristic algorithm which achieves better classification performance and select more relevant genes across various datasets. Furthermore, several studies have extended incremental algorithms to address challenges posed by incomplete dynamic decision tables. For instance, Giang et *al.* (2021) designed hybrid incremental algorithms utilizing tolerance sets to manage missing data within dynamic decision tables. However, the effectiveness of attribute reduction methods built on the FRS model is often compromised when dealing with high-noise datasets or cases of low classification accuracy, as demonstrated by Hung and Yang (2007). These shortcomings have motivated researchers to explore more robust alternatives.

In recent studies, the intuitionistic fuzzy rough set (IFRS) model has been investigated as a means to improve attribute reduction process. By incorporating non- membership functions, IFRS enhances the system's ability to manage noise, enabling better classification of uncertain objects (Huang et *al.* 2014). As a result, IFRS-based reduction algorithms have shown superior robustness and effectiveness compared to traditional FRS-based methods, particularly in noisy or low-performance scenarios. Singh *et al.* (2020) defined novel mechanisms for attribute reduction in incomplete information system within the framework of IFRS, including the definition of intuitionistic fuzzy tolerance relation and development of lower and upper approximations operators. They also presented a greedy algorithm and a practical example to illustrate their approach. Nguyen *et al.* (2021) proposed a distance measure in the intuitionistic fuzzy set (IFS) model and introduced IFDBAR algorithm for reduct computation in decision tables. However, one limitation of IFS-based attribute reduction method particularly employing filter strategies, is their relatively slow processing speed.

The rough neutrosophic set, introduced by Broumi *et al*. (2014) was developed to address vagueness, uncertainty, imprecise, inconsistent and incomplete present in datasets. Given that human decision-making often depends on expert judgement which can be unreliable. This set offers to capture and process imperfect knowledge. Despite its potential, the integration of rough set with neutrosophic set theory has seen limited exploration in the field of attribute reduction. (Tiwari *et al.* 2018) claimed that novel approach of attribute reduction employs a tolerance-based intuitionistic fuzzy rough set approach. In their work, they defined lower and upper approximations based on tolerance relations and calculated the degree of dependency of decision attributes on the set of conditional attributes. Their algorithm was applied to a sample data set and compared the results with tolerance-based fuzzy rough set method. Ab Ghani *et al.* (2022) proposed approximation constructs for rough neutrosophic set using tolerance relations. Their research emphasized type-1 of two granulation and multi-granulation models such as T1-TGRNS and T1-MGRNS and explored their properties in terms of lower and upper approximations. Wang *et al.* (2023) proposed a multi-scale single-valued neutrosophic system utilizing a dominance relation, which leverages relative distance preference degrees. They detailed the procedures for computing test and decision costs using a rough set-based risk cost formula and identified optimal scale combinations that minimize total system cost.

This paper introduces a new method that combines tolerance-based relations with the rough neutrosophic set for attribute reduction. First, it introduces novel formulations for lower and upper approximations by replacing the traditional indiscernibility relation with a tolerance relation on similarity between two objects. Secondly, an algorithm is proposed to identify the reduct set within a rough neutrosophic decision system. The proposed method effectively addresses vagueness, uncertainty and incompleteness by integrating the rough set model with the neutrosophic set model. In addition, the proposed algorithm evaluates the degree of dependency between condition and decision attributes to derive a minimal reduct. An illustrative example is provided using coastal erosion data to demonstrate the method practically and improved performance.

2. Rough Neutrosophic Sets

In order to build the theoretical foundation required for the development of tolerance-based rough neutrosophic framework. This section reviews basic definitions of neutrosophic sets, such as single-valued neutrosophic relations and rough neutrosophic sets.

Definition 2.1. (Smarandache 2006) Let X be a universal set. A neutrosophic set V defined on X is expressed as:

$$V = \{\langle x, T_V(x), I_V(x), F_V(x)\rangle : x \in X\} \tag{1}$$

The function T_V, I_V, F_V are real-valued mappings from X to the extended intervals $]^{-}0{:}\,1^{+}[$, representing the degrees of truth, indeterminacy, and falsity respectively. There are no constraints on the sum of these values from any element $x \in X$.

Neutrosophic set offer a flexible and comprehensive structure for capturing data characterized by inconsistency, imprecision and incomplete information. The truth and falsity components relate to how strongly an element belongs or does not belong to the set, whereas the indeterminacy degree quantifies uncertainty independently of the other two. All the degrees fall within the standard interval $[0,1]$.

Definition 2.2. (Yang *et al.* 2016) A single valued neutrosophic relation (SVNS) $\mathcal{R}$ on the Cartesian product $U \times U$ defined as:

$$\mathcal{R} = \{\langle (x,y), \eta_{\mathcal{R}}(x,y), \beta_{\mathcal{R}}(x,y), \upsilon_{\mathcal{R}}(x,y)\rangle | (x,y) \in U \times U\} \tag{2}$$

The functions $\eta_{\mathcal{R}}, \beta_{\mathcal{R}}$, and $\upsilon_{\mathcal{R}}$ map from $U \times U$ to the interval $[0,1]$ representing the truth membership, indeterminacy membership and falsity-membership degrees respectively, for each pair of (x,y) in the universe U.

Definition 2.3. A rough neutrosophic binary relation between objects $x_i, x_j \in U$, denoted by $\langle \eta_V(x_i,x_j), \beta_V(x_i,x_j), \upsilon_V(x_i,x_j)\rangle$ $x_i, x_j \in U$ is classified as a rough neutrosophic tolerance relation if it satisfies the properties of reflexive and symmetric.

Definition 2.4. (Ab Ghani *et al.* 2022) Let $\mathbb{S} = \langle U, A\rangle$ represent a rough neutrosophic information system and let F be neutrosophic set defined over the universe U, where $a \subseteq A$.

The lower and upper approximation of $F \subseteq U$ with respect to a covering based on tolerance relation are defined as follows:

$$\mathcal{L}(F) := \bigcup \{[x]^F_{tol(\alpha)} : x \in U, [x]^F_{tol(\alpha)} \subseteq F \wedge y \in [x]^F_{tol(\alpha)}\}$$
$$= \{< x \in U, \eta_{\mathcal{L}(F)}(x), \beta_{\mathcal{L}(F)}(x), \nu_{\mathcal{L}(F)}(x) >: \forall y \in U, \langle x, y\rangle \in tol(\alpha) \quad (3)$$

$$\mathcal{U}(F) := \bigcup \{[x]^F_{tol(\alpha)} : x \in U, [x]^F_{tol(\alpha)} \cap F \neq \emptyset\}$$
$$= \{< x \in U, \eta_{\mathcal{U}(F)}(x), \beta_{\mathcal{U}(F)}(x), \nu_{\mathcal{U}(F)}(x) >: \exists y \in U, \langle x, y\rangle \in tol(\alpha) \quad (4)$$

Definition 2.5. (Broumi *et al.* 2014) Let U be a non-empty set and let Q denote a tolerance relation defined on U. Suppose V is a neutrosophic set on U characterized by three membership functions which are η_v is the truth-membership function, β_v is the indeterminacy-membership function, and ν_v is the falsity-membership function. The lower and upper approximations of V under the approximation (U,Q) denoted by $\overline{N}(V)$ and $\underline{N}(V)$ respectively, are defined as follows:

$$\overline{N}(\mathrm{V}) = \{\langle x, \eta_{\overline{N}(\mathrm{V})}(x), \beta_{\overline{N}(\mathrm{V})}(x), \nu_{\overline{N}(\mathrm{V})}(x)\rangle | y \in [x]_Q, x \in U\} \quad (5)$$

$$\underline{N}(\mathrm{V}) = \{\langle x, \eta_{\underline{N}(\mathrm{V})}(x), \beta_{\underline{N}(\mathrm{V})}(x), \nu_{\underline{N}(\mathrm{V})}(x)\rangle | y \in [x]_Q, x \in U\} \quad (6)$$

where

$$T_{\overline{N}(\mathrm{V})}(x) = \vee_{y \in [x]_Q} T_V(y), I_{\overline{N}(V)}(x) = \wedge_{y \in [x]_Q} I_V(y), F_{\overline{N}(V)}(x) = \wedge_{y \in [x]_Q} F_V(y),$$

$$T_{\underline{N}(V)}(x) = \wedge_{y \in [x]_Q} T_V(y), I_{\underline{N}(V)}(x) = \vee_{y \in [x]_Q} I_V(y), F_{\underline{N}(V)}(x) = \vee_{y \in [x]_Q} F_V(y)$$

The symbol $\vee$ and $\wedge$ denote the maximum and minimum operators across the tolerance class $[x]_Q$. The functions $T_V(y)$, $I_V(y)$, and $F_V(y)$ represent the truth-membership, indeterminacy-membership, and falsity-membership of element y in the neutrosophic set V. These degrees are bounded within the interval $[0,1]$, ensuring the following conditions hold:

$$0 \leq T_{\underline{N}(V)}(x) + I_{\underline{N}(V)}(x) + F_{\underline{N}(V)}(x) \leq 3$$

and

$$0 \leq T_{\overline{N}(V)}(x) + I_{\overline{N}(V)}(x) + F_{\overline{N}(V)}(x) \leq 3$$

The lower and upper rough neutrosophic approximation operators are denoted as $\underline{N}(V)$ and $\overline{N}(V)$. The pair of $\underline{N}(V), \overline{N}(V)$ defines the rough neutrosophic set in (U, Q). Further logical properties and structural characteristics of such sets have been explored by Zainal *et al.* (2021).

Definition 2.6. A rough neutrosophic information system $(RNIS)$ is formally defined as a quadruple $S = (U, C \cup D, V, f)$, where U is called universe consisting of a non-empty and finite set of objects, C represents a finite conditional attribute, $D = \{d\}$ is a single set containing the decision attribute d, with the condition that $C \cap D = \emptyset$, V denotes the set of all rough neutrosophic values, composed of $V = V_1 \cup V_2$, where V_1 and V_2 are the value domains corresponding to the condition and decision attribute, respectively. The information function f is a map from $U \times (C \cup D)$ onto V, such that $f(x, c)$ and $f(x, d)$ are rough neutrosophic values, represented as $f(x, c) = \langle \eta_c(x), \beta_c(x), v_c(x) \rangle$ and $f(x, d) = \langle \eta_d(x), \beta_d(x), v_d(x) \rangle$. Thus, $f(x, c)$ refers to the rough neutrosophic value of object x under the condition attribute c, while $f(x, d)$ represent rough represents rough neutrosophic of x under decision attribute d.

Example 1 A rough neutrosophic decision system is illustrated in Table 1, where the universe is given by $U = \{x_1, x_2, \dots, x_6\}$ and the set of conditional attributes is $C = \{c_1, c_2, c_3, c_4, c_5\}$. Each object is described by rough neutrosophic values for the conditional attributes, and a corresponding decision class denoted by d.

Table 1: Rough neutrosophic decision system

U	c_1	c_2	c_3	c_4	c_5	d
x_1	⟨0.2,0.4,0.2⟩	⟨0.1,0.7,0.3⟩	⟨0.2,0.6,0.1⟩	⟨0.4,0.6,0.2⟩	⟨0.2,0.8,0.2⟩	1
x_2	⟨0.1,0.7,0.3⟩	⟨0.1,0.8,0.3⟩	⟨0.3,0.6,0.1⟩	⟨0.5,0.2,0.3⟩	⟨0.2,0.7,0.2⟩	2
x_3	⟨0.1,0.8,0.3⟩	⟨0.1,0.8,0.3⟩	⟨0.2,0.8,0.2⟩	⟨0.5,0.4,0.2⟩	⟨0.4,0.6,0.2⟩	1
x_4	⟨0.1,0.9,0.1⟩	⟨0.6,0.3,0.3⟩	⟨0.2,0.7,0.2⟩	⟨0.2,0.8,0.2⟩	⟨0.4,0.6,0.2⟩	1
x_5	⟨0.4,0.6,0.3⟩	⟨0.2,0.6,0.1⟩	⟨0.2,0.8,0.2⟩	⟨0.2,0.8,0.2⟩	⟨0.2,0.8,0.2⟩	2
x_6	⟨0.4,0.6,0.2⟩	⟨0.2,0.6,0.1⟩	⟨0.2,0.8,0.2⟩	⟨0.2,0.4,0.3⟩	⟨0.2,0.8,0.2⟩	1

In practical applications, decision systems often consist of high-dimensional data, which can pose challenges for analysis and computation. To address this, the following sections introduce rough neutrosophic techniques for dimensionality reduction by employing feature selection techniques aimed at identifying the most informative attributes.

3. Tolerance-Based Rough Neutrosophic Set Approach for Attribute Reduction

In this section, we define the similarity between pairs of objects based on attributes or subsets of attributes. To enhance classification accuracy, we introduce a tolerance relation within the rough neutrosophic set (*RNS*) framework, incorporating threshold values applied to membership, indeterminacy, and non-membership values.

Rough neutrosophic decision system (*RNDS*) relies on equivalence relations to partition of U, which can limit the effectiveness of knowledge discovery. To address this constraint, we propose a similarity degree that captures the similarity degree of rough neutrosophic values. Inspired by the similarity model proposed by Feng and Li (2013), this work generalizes the concept to the tolerance-based rough neutrosophic environment, as detailed in the following.

Definition 3.1. Assume $RNDS = (U, C \cup D, V, f)$, for any $x_i, x_j \in U, c_k \in C$, the two rough neutrosophic values $f(x_i, c_k) = \langle \eta_{c_k}(x_i), \beta_{c_k}(x_i), v_{c_k}(x_i) \rangle$ and $f(x_j, c_k) = \langle \eta_{c_k}(x_j), \beta_{c_k}(x_j), v_{c_k}(x_j) \rangle$, the similarity degree between x_i and x_j with respect to attribute c_k is defined using a weighted Euclidean distance as follows:

$$sim_{c_k}(x_i, x_j) = 1 - \sqrt{\alpha\left(\eta_{c_k}(x_i) - \eta_{c_k}(x_j)\right)^2 + \omega\left(\beta_{c_k}(x_i) - \beta_{c_k}(x_j)\right)^2 + \gamma\left(v_{c_k}(x_i) - v_{c_k}(x_j)\right)^2} \quad (7)$$

where α, ω, γ are weighting factors and $\eta_{c_k}(x_i)$, $\beta_{c_k}(x_i)$ and $v_{c_k}(x_i)$ are membership, indeterminacy, and non-membership of an object x_i.

Remark 3.1. The parameter values in $RNDS$ are computed to represent object characteristics are determined based on user requirements, subject to the following constraints:

1. $\eta \geq v > \beta$
2. $0 \leq \eta, v, \beta \leq 1$
3. $\eta + \beta + v = 1$

Property 3.1

Let $RNDS = (U, C \cup D, V, f)$ for any $x_i, x_j \in U, a \in C$, the following condition holds:

1. $0 \leq sim_a(x_i, x_j) \leq 1$
2. $sim_a(x_i, x_j) = sim_a(x_j, x_i)$
3. $f(x_i, a) = f(x_j, a) \Leftrightarrow sim_a(x_i, x_j) = 1$
4. Consider $f(x_i, a) = \langle 1,0 \rangle$, $f(x_j, a) = \langle 0,1 \rangle$ and let $\alpha = \omega = 0.5$, Under these conditions, the similarity measure is $sim_a(x_i, x_j) = 0$ indicating that x_i and x_j are completely dissimilar in term of a.

Definition 3.2 Let $RNDS = (U, C \cup D, V, f)$ be a rough neutrosophic decision system and let $A \subseteq C$ be a subset of conditional attributes. For threshold $\delta \in [0,1]$, a δ- similarity relation $\mathcal{T}^{\delta}$ in the $RNDS$ is defined as:

$$\mathcal{T}^{\delta}(A) = \{(x_i, x_j) \in U \times U: sim_a(x_i, x_j) \geq \delta, for\ all\ a \in A\}$$

This relation $\mathcal{T}^{\delta}$ is reflexive and symmetric, but not transitive.

To evaluate the similarity relation between two objects with respect to a $M \subseteq C$, the relation is extended as follows:

$$(x_i, x_j) \in sim_M^{\delta} \text{ if and only if } \prod_{f \in M} sim_f\,(x_i, x_j) \geq \delta \quad (8)$$

δ is similarity threshold where $\delta \in [0,1]$. It determines the required degree of similarity for inclusion under the tolerance relation.

Definition 3.3 Given the similarity relation sim_M^{δ}, the tolerance relation class of an object $x_i \in U$ is defined as:

$$sim_M^{\delta} = \{x_j \in U | (x_i, x_j) \in sim_M^{\delta}\} \quad (9)$$

This class includes all objects x_j that are sufficiently similar to x_i under the specified attribute subset M and threshold δ.

Definition 3.4 Given a knowledge representation system $RNDS = (U, C \cup D, V, f)$, and let $X \subseteq U$ be lower and upper approximation respectively are

$$\mathcal{L}_M^{\delta}(F) = \{x_i : SIM_M^{\delta}(x_i) \subseteq X\}$$
$$= \{\langle x \in U, \eta_{\underline{N}(\mathrm{V})}(x), \beta_{\underline{N}(\mathrm{V})}(x), \upsilon_{\underline{N}(\mathrm{V})}(x)\rangle : y \in U, \langle x, y\rangle \in tol(\alpha)\}, \quad (10)$$

$$\mathcal{U}_M^{\delta}(F) = \{x_i : SIM_M^{\delta}(x_i) \cap X \neq \phi\}$$
$$= \{\langle x \in U, \mu_{\overline{N}(\mathrm{V})}(x), \upsilon_{\overline{N}(\mathrm{V})}(x), \omega_{\overline{N}(\mathrm{V})}(x)\rangle : y \in U, \langle x, y\rangle \in tol(\alpha)\}. \quad (11)$$

The pair $\left(\mathcal{L}_M^{\delta}(F), \mathcal{U}_M^{\delta}(F)\right)$ is referred to as a rough neutrosophic toleration set. These approximations follow the same conceptual foundation as in the classical RNS. Let Z be a set of attributes generating the tolerance relation over U.

Definition 3.5 Assume $RNDS = (U, C \cup D, V, f)$. The Z-positive region with respect to M, denoted by

$$pos_M^{\delta}(Z) = \cup_{X \in U/Z} \mathcal{L}_M^{\delta}(F) \quad (12)$$

where $Z \subseteq C$ and $Z = \{d\}$.

Definition 3.6 The dependency degree Z to $M$is determined by the ratio of $pos_M^{\delta}(Z)$ to $|U|$:

$$\Gamma_M^{\delta}(Z) = \frac{|pos_M^{\delta}(Z)|}{|U|} \quad (13)$$

The role of $\Gamma_M^{\delta}(Z)$, serves as key indicator of how well the attribute set M, which is as important as attributes M in approximating decision Z. During the attribute reduction process, attributes are added incrementally to the current subset, and at each step, the change of degree of dependency is evaluated. If including an additional attribute does not increase the dependency degree, the process is terminated, and the resulting subset is identified as the reduct. The formulation extends Pawlak's dependency model from classical rough sets to RNS; thereby generalizing it to support more complex, uncertain environments over arbitrary universe.

Theorem 3.1. *Let $\left(U, C \cup D, V_{RNS,} RNS\right)$ be RNDS. Let $Z \subseteq C$ and $X \subseteq U$, then $\mathcal{L}_M^{\delta}(F) \subseteq X \subseteq \mathcal{U}_M^{\delta}(F)$.*

Proof.
Considering that $y \in sim_M^{\delta}(y)$. Hence, $y \in X$ producing $\mathcal{L}_M^{\delta}(F) \subseteq X$. Assume $y \in X$. From the point $y \in sim_M^{\delta}(y) \longrightarrow |sim_M^{\delta}(y) \cap X \neq \phi \longrightarrow y \in \mathcal{U}_M^{\delta}(F) \longrightarrow X \longrightarrow \mathcal{U}_M^{\delta}(F)$, $\mathcal{L}_M^{\delta}(F) \subseteq X \subseteq \mathcal{U}_M^{\delta}(F)$ is produced. □

Theorem 3.2. *Assume $\left(U, C \cup D, V_{RNS,} RNS\right)$ be an RNDS. Let $M_1 \subseteq M_2 \subseteq C$ and $X \subseteq U$, then*

1. $\mathcal{L}_{M_1}^{\delta}(F) \subseteq \mathcal{L}_{M_2}^{\delta}(F)$
2. $\mathcal{U}_{M_1}^{\delta}(F) \subseteq \mathcal{U}_{M_2}^{\delta}(F)$

Proof.

1. Assume $y \in \mathcal{L}_{M_1}^{\delta}(F)$ then, $sim_{M_1}^{\delta}(y) \subseteq X$, So $M_1 \subseteq M_2 \longrightarrow sim_{M_2}^{\delta}(y) \subseteq sim_{M_1}^{\delta}(y)$. Hence, $sim_{M_2}^{\delta}(y) \subseteq X \longrightarrow y \in \mathcal{L}_{M_2}^{\delta}(F)$. Therefore, $\mathcal{L}_{M_1}^{\delta}(F) \subseteq \mathcal{L}_{M_2}^{\delta}(F)$.

2. Assume $y \in \mathcal{U}_{M_2}^{\delta}(F)$, then $sim_{M_2}^{\delta} \cap X \neq \phi$, So $M_1 \subseteq M_2 \longrightarrow sim_{M_2}^{\delta}(y) \subseteq sim_{M_1}^{\delta}(y)$ Hence, $sim_{M_1}^{\delta} \cap X \neq \phi \longrightarrow y \in \mathcal{U}_{M_1}^{\delta}(F)$. Therefore, $\mathcal{U}_{M_2}^{\delta}(F) \subseteq \mathcal{U}_{M_1}^{\delta}(F)$. □

Theorem 3.3 *Assume* $(U, C \cup D, V_{RNS,} RNS)$ *be an RNDS. Let* $M \subseteq C, \delta_1 \leq \delta_2$ *and* $X \subseteq U$, *then*

1. $\mathcal{L}_M^{\delta_1}(F) \subseteq \mathcal{L}_M^{\delta_2}(F)$
2. $\mathcal{U}_M^{\delta_1}(F) \subseteq \mathcal{U}_M^{\delta_2}(F)$

Proof.

1. Assume $\mathcal{L}_M^{\delta_1}(F)$, then $sim_M^{\delta_1}(y) \subseteq X$. If $z \in sim_M^{\delta_2}(y)$, then $(y, z) \in sim_M^{\delta_2}(y) \leftrightarrow \prod_{a \in M} sim_a\,(y, z) \geq \delta_2 \leftrightarrow \prod_{a \in M} sim_a\,(y, z) \geq \delta_1 \delta_2 \geq \delta_1 \leftrightarrow (y, z) \in sim_M^{\delta_1}(y) \leftrightarrow z \in sim_M^{\delta_1}(y) \rightarrow sim_M^{\delta_2}(y) \subseteq sim_M^{\delta_1}(y) \rightarrow sim_M^{\delta_2} \subseteq X \longrightarrow y \in \mathcal{L}_M^{\delta_2}(F)$. Therefore, $\mathcal{L}_M^{\delta_1}(F) \subseteq \mathcal{L}_M^{\delta_2}(F)$
2. Assume $\mathcal{U}_M^{\delta_2}(F)$, then $sim_M^{\delta_2}(y) \cap X \neq \phi. sim_M^{\delta_2}(y) \subseteq sim_M^{\delta_1}(y) \rightarrow sim_M^{\delta_1}(y) \cap X \neq \phi \rightarrow y \in \mathcal{U}_M^{\delta_1}(F)$. Therefore, $\mathcal{U}_M^{\delta_2}(F) \subseteq \mathcal{U}_M^{\delta_1}(F)$. □

4. Algorithm for Tolerance-Based Rough Neutrosophic Reduction

This section presents an attribute reduction algorithm designed to identify a reduct based on tolerance relation and considering the degree of dependency denoted by $\Gamma_M^{\delta}(Z)$. The algorithm begins with a null set and sequentially incorporated attributes, seeking the maximum increase in dependency on the decision attribute. This process produces the highest values for the dataset, with a value of 1 in the case of consistent system. This algorithm eliminates exhaustive evaluation of all potential subsets of conditional attributes owing to the minimal reduct property of the decision system. The proposed reduct algorithm proceeds as follows:

Step 1: Construct the original information system into RNS decision table as illustrated in Example 1.

Step 2: Calculate the rough neutrosophic tolerance relation between each pair of objects based conditional attributes as defined in Definition 3.1.

Step 3: Determine the lower and upper approximation of set using RNS tolerance classes corresponding to identical decision classes, as defined in Definition 3.2.

Step 4: Compute the positive region and degree of dependency of d for each conditional attribute.

Step 5: Select the attribute that yields the highest condition attribute for degree of dependency; this attribute is designated as the first reduct.

Step 6: Add other attributes to obtain a new reduct set and calculate the degree of dependency of conditional attributes.

Step 7: Repeat Step 5 and 6. If the inclusion of futher attributes does not lead to an increase in the degree of dependency compared to the previous iteration. The process is halted and the current set is declared the final reduct.

5. Illustrative Example

The key factors contributing to coastal erosion are adapted from the study by (Luo *et al.* 2013). Based on their findings, six significant attributes or condition of coastal erosion are storm surge (c_1), hydrodynamic wave and current (c_2), imbalance sediment supply(c_3), sea level rise(c_4) and sand mining activities (c_5).

Table 2 outlines the coastal erosion decision system where the universe is defined as $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and the corresponding conditional attribute set is given by $C = \{c_1, c_2, \ldots, c_6\}$.

Table 2: Coastal erosion decision system

U	c_1	c_2	c_3	c_4	c_5	c_6	d
x_1	⟨0.32,0.48,0.20⟩	⟨0.32,0.48,0.20⟩	⟨0.80,0.00,0.10⟩	⟨0.64,0.16,0.30⟩	⟨0.32,0.48,0.20⟩	⟨0.16,0.64,0.30⟩	1
x_2	⟨0.48,0.32,0.20⟩	⟨0.80,0.00,0.10⟩	⟨0.48,0.32,0.20⟩	⟨0.64,0.16,0.30⟩	⟨0.16,0.64,0.30⟩	⟨0.16,0.64,0.30⟩	0
x_3	⟨0.64,0.16,0.30⟩	⟨0.32,0.48,0.20⟩	⟨0.32,0.48,0.20⟩	⟨0.48,0.32,0.20⟩	⟨0.80,0.00,0.10⟩	⟨0.80,0.00,0.10⟩	1
x_4	⟨0.80,0.00,0.10⟩	⟨0.64,0.16,0.30⟩	⟨0.16,0.64,0.30⟩	⟨0.80,0.00,0.10⟩	⟨0.48,0.32,0.20⟩	⟨0.32,0.48,0.20⟩	0
x_5	⟨0.16,0.64,0.30⟩	⟨0.80,0.00,0.10⟩	⟨0.64,0.16,0.30⟩	⟨0.32,0.48,0.20⟩	⟨0.32,0.48,0.20⟩	⟨0.48,0.32,0.20⟩	0
x_6	⟨0.48,0.32,0.20⟩	⟨0.48,0.32,0.20⟩	⟨0.64,0.16,0.30⟩	⟨0.16,0.64,0.30⟩	⟨0.64,0.16,0.30⟩	⟨0.64,0.16,0.30⟩	1

Step 1: Begin by constructing the coastal erosion decision system, as illustrated in Table 2.

Step 2: Calculate the rough neutrosophic tolerance relation between two objects concerning conditional attributes. Utilize a tolerance based RNS for attribute selection. The transformed decision system is obtained using indeterminacy as 0.2 (refer to Table 2). The decision classes of decision systems are defined as follows:

$$U/Z=\{\{x_1, x_3, x_6\}, \{x_2, x_4, x_5\}\}.$$

Based on the weighting factors $\alpha = 0.4, \omega = 0.2, \gamma = 0.4$, and $\delta = 0.8$, the tolerance class for attribute set c_1 is measured using the similarity measure:

$$U/sim_{c_1}^{\delta}= \{\{x_1, x_2, x_6\}, \{x_1, x_5\}, \{x_2, x_3, x_6\}, \{x_3, x_4\}\}.$$

Similarly, several tolerance classes for other attributes are expressed as

$$U/sim_{c_2}^{\delta}= \{\{x_1, x_3, x_6\}, \{x_2, x_4, x_5\}\}$$

$$U/sim_{c_3}^{\delta}= \{\{x_1, x_5, x_6\}, \{x_2, x_3\}, \{x_2, x_5, x_6\}, \{x_3, x_4\}\}$$

$$U/sim_{c_4}^{\delta}= \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_3, x_5\}, \{x_5, x_6\}\}$$

$$U/sim_{c_5}^{\delta}=\{\{x_1, x_2\}, \{x_1, x_4, x_5\}, \{x_3, x_6\}, \{x_4, x_6\}\}$$

$$U/sim_{c_6}^{\delta}=\{\{x_1, x_2, x_4\}, \{x_3, x_5\}, \{x_4, x_5\}\{x_5, x_6\}\}$$

Step 3: The lower approximation of decision classes for attribute set $c_1$are:

$$\mathcal{L}^{\delta}_{c_1}(\text{F})\ \{1,3,6\} = \{x_i | sim^{\delta}_{c_1}(x_i) \subset \{1,3,6\}\} = \emptyset$$

$$\mathcal{L}^{\delta}_{c_1}(F)\ \{2,4,5\} = \{x_i | sim^{\delta}_{c_1}(x_i) \subset \{2,4,5\}\} = \emptyset$$

Step 4: The positive region is calculated, resulting in $pos^{\delta}_{c_1}(Z) = \emptyset \cup \emptyset = \emptyset$.

Step 5: The degree of dependency is obtained as:

$$\Gamma^{\delta}_{c_1}(Z) = \frac{0}{6}$$

Step 6: For the other attributes, the degree of dependencies are :

$$\Gamma^{\delta}_{c_2}(Z) = \frac{6}{6}$$
$$\Gamma^{\delta}_{c_3}(Z) = \frac{0}{6}$$
$$\Gamma^{\delta}_{c_4}(Z) = \frac{0}{6}$$
$$\Gamma^{\delta}_{c_5}(Z) = \frac{2}{6}$$
$$\Gamma^{\delta}_{c_6}(Z) = \frac{4}{6}$$

Step 7: Since the degree of dependency has reached its maximum possible value of 1, no further improvement can be achieved by adding other attributes. As a result, the algorithm terminates. The final reducts obtained as c_2 and c_2 as hydronamic wave and current, respectively.

6. Conclusion

This paper introduces an innovative tolerance-based rough neutrosophic set approach for attribute reduction. The approach defines lower and upper approximations using a threshold value δ and introduces the method for calculating the degree of dependency of decision attribute with respect to subset of conditional attributes within the framework of tolerance-based rough neutrosophic set. Furthermore, the theoretical foundations of the model are established though the proof of relevant theorems concerning the approximation operators. To validate the approach, the algorithm was applied to dataset which results indicating its effectiveness in identifying minimal reducts in decision systems. Moreover, by turning the parameter δ, the model can be adapted to better tolerate noise or handle faults in real-world data. Our proposed algorithm is also capable of managing uncertainty, incompleteness and vagueness in decision system.

The proposed approach offers significant potential for selecting the most non redundant features in machine learning, thereby enhancing data precision in intelligent system applications. For future work, we intend to extend this model into variable precision rough neutrosophic set model to address high levels of fault in a data by losing the membership in lower approximation and focus on upper approximation. Besides, we plan to explore its application in incomplete decision system, particularly for set-valued data to further enhance its robustness and adaptability in tolerance rough neutrosophic set.

Acknowledgments

We acknowledge the support from Talent and Publication Enhancement Research Grant (TAPE-RG), Grant No. 55226, Universiti Malaysia Terengganu.

References

Ab Ghani A.T., Abdullah L. & Peng N. 2022. On type-1 granulation and approximation of tolerance based rough neutrosophic sets. *Proceedings of the 8th International Conference on Computational Science and Technology: ICCST 2021*, pp. 171-182.

Ahmad N., Berg D. & Simons G.R. 2006. The integration of analytical hierarchy process and data envelopment analysis in a multi-criteria decision-making problem. *International Journal of Information Technology & Decision Making* **5**(2): 263–276.

Alfares H.K. & Duffuaa S.O. 2008. Determining aggregate criteria weights from criteria rankings by a group of decision makers. *International Journal of Information Technology & Decision Making* **7**(4): 769–781.

Atanassov K.T. 1999. *Intuitionistic Fuzzy Sets*. Heidelberg, Germany: Physica-Verlag HD.

Baser F., Koc O. & Selcuk-Kestel A.S. 2023. Credit risk evaluation using clustering based fuzzy classification method. *Expert Systems with Applications* **223**: 119882.

Bonanno D., Nock K., Smith L., Elmore P. & Petry F. 2017. An approach to explainable deep learning using fuzzy inference. *Proc. SPIE 10207, Next-Generation Analyst V*, pp. 132-136.

Broumi S., Smarandache F. & Dhar M. 2014. Rough neutrosophic sets. *Neutrosophic Sets and Systems* **3**: 60–65.

Chebrolu S. & Sanjeevi S.G. 2017. Attribute reduction on real-valued data in rough set theory using hybrid artificial bee colony: extended FTSBPSD algorithm. *Soft Computing* **21**(22): 7543–7569.

Dubois D. & Prade H. 1990. Rough fuzzy sets and fuzzy rough sets. *International Journal of General System* **17**(2-3): 191-209.

Feng Q. & Li R. 2013. Discernibility matrix based attribute reduction in intuitionistic fuzzy decision systems. In *Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing: 14th International Conference, RSFDGrC 2013,* pp. 147-156.

Giang N.L., Tuan N.A., Ngan T.T., Son N.N. & Thang N.T. 2021. Filter-wrapper incremental algorithms for finding reduct in incomplete decision systems when adding and deleting an attribute set. *International Journal of Data Warehousing and Mining* **17**(2): 39-62.

Gong Z., Sun B. & Chen D. 2008. Rough set theory for the interval-valued fuzzy information systems. *Information Sciences* **178**(8): 1968-1985.

Gou X., Xu Z., Liao H. & Herrera F. 2021. Probabilistic double hierarchy linguistic term set and its use in designing an improved VIKOR method: The application in smart healthcare. *Journal of the Operational Research Society* **72**(12): 2611-2630.

Han J., Yang C., Lim C.C., Zhou X., Shi P. & Gui W. 2020. Power scheduling optimization under single-valued neutrosophic uncertainty. *Neurocomputing* **382**: 12-20.

Huang B., Guo C.X., Zhuang Y.L., Li H.X. & Zhou X.Z. 2014. Intuitionistic fuzzy multigranulation rough sets. *Information Sciences* **277**: 299-320.

Hung W.L. & Yang M.S. 2007. Similarity measures of intuitionistic fuzzy sets based on Lp metric. *International Journal of Approximate Reasoning* **46**(1): 120-136.

Jensen R. & Shen Q. 2004. Semantics-preserving dimensionality reduction: Rough and fuzzy-rough-based approaches. *IEEE Transactions on Knowledge and Data Engineering* **16**(12): 1457-1471.

Kondo M. 2006. On the structure of generalized rough sets. *Information Sciences* **176**(5): 589-600.

Liang B., Jin E., Wei L. & Hu R. 2024. Knowledge granularity attribute reduction algorithm for incomplete systems in a clustering context. *Mathematics* **12**(2): 333.

Lin T.Y. 1992. Topological and fuzzy rough sets. In Słowiński R (ed.). *Intelligent Decision Support: Handbook of Applications and Advances of the Rough Sets Theory*: 287-304. Dordrecht: Springer-Science+Business Media, B.V.

Luo S., Wang H. & Cai F. 2013. An integrated risk assessment of coastal erosion based on fuzzy set theory along Fujian coast, southeast China. *Ocean & Coastal Management* **84**: 68-76.

Nguyen T.T., Giang N.L., Tran D.T., Nguyen T.T., Nguyen H.Q., Pham A.V. & Vu T.D. 2021. A novel filter-wrapper algorithm on intuitionistic fuzzy set for attribute reduction from decision tables. *International Journal of Data Warehousing and Mining* **17**(4): 67-100.

Pattanayak R.M., Behera H.S. & Panigrahi S. 2021. A novel probabilistic intuitionistic fuzzy set based model for high order fuzzy time series forecasting. *Engineering Applications of Artificial Intelligence* **99**: 104136.

Pawlak Z. 1982. Rough sets. *International Journal of Computer & Information Sciences* **11**: 341-356.

Qian Y., Liang J. & Dang C. 2008. Interval ordered information systems. *Computers & Mathematics with Applications* **56**(8): 1994-2009.

Şahin R. & Karabacak M. 2020. A novel similarity measure for single-valued neutrosophic sets and their applications

in medical diagnosis, taxonomy, and clustering analysis. In Smarandache F. (ed.). *Optimization Theory Based on Neutrosophic and Plithogenic Sets*: 315–341. London; San Diego, CA: Academic Press.
Singh S., Shreevastava S. & Som T. 2020. Attribute reduction of incomplete information systems: an intuitionistic fuzzy rough set approach. *International Conference on Information Technology and Applied Mathematics*, pp. 628-643.
Skowron A. & Rauszer C. 1992.The discernibility matrices and functions in information systems. In Słowiński R. (ed.). *Intelligent Decision Support. Theory and Decision Library*: 331-362. Dordrecht, NL: Springer.
Smarandache, F. (2006, May). Neutrosophic set-a generalization of the intuitionistic fuzzy set. In *2006 IEEE international conference on granular computing* (pp. 38-42). IEEE.
Su, N., An, X., Yan, C., & Ji, S. (2020). Incremental attribute reduction method based on chi-square statistics and information entropy. *IEEE Access*, *8*, 98234-98243.
Sun, L., Wang, L., Ding, W., Qian, Y., & Xu, J. (2020). Feature selection using fuzzy neighborhood entropy-based uncertainty measures for fuzzy neighborhood multigranulation rough sets. *IEEE Transactions on Fuzzy Systems*, *29*(1), 19-33.
Tiwari, A. K., Shreevastava, S., Som, T., & Shukla, K. K. (2018). Tolerance-based intuitionistic fuzzy-rough set approach for attribute reduction. *Expert Systems with Applications*, *101*, 205-212.
Vasilakakis, M. D., & Iakovidis, D. K. (2023). Fuzzy similarity phrases for interpretable data classification. *Information Sciences*, *624*, 881-907.
Wang, W., Huang, B., & Wang, T. (2023). Optimal scale selection based on multi-scale single-valued neutrosophic decision-theoretic rough set with cost-sensitivity. *International Journal of Approximate Reasoning*, *155*, 132-144.
Yang, H. L., Guo, Z. L., She, Y., & Liao, X. (2016). On single valued neutrosophic relations. *Journal of Intelligent & Fuzzy Systems*, *30*(2), 1045-1056.
Yazdani, M., Torkayesh, A. E., Stević, Ž., Chatterjee, P., Ahari, S. A., & Hernandez, V. D. (2021). An interval valued neutrosophic decision-making structure for sustainable supplier selection. *Expert Systems with Applications*, *183*, 115354.
Yuan, Z., Chen, H., Li, T., Liu, J., & Wang, S. (2021). Fuzzy information entropy-based adaptive approach for hybrid feature outlier detection. *Fuzzy Sets and Systems*, *421*, 1-28.
Zadeh, L. A. (1965). Zadeh, fuzzy sets. *Inform Control*, *8*, 338-353.
Zainal, S. M., Ab Ghani, A. T., & Abdullah, L. (2021, May). On some logical relation properties for rough neutrosophic set. In *Journal of Physics: Conference Series* (Vol. 1878, No. 1, p. 012048). IOP Publishing.
Zhang, M., Zhang, L., & Cheng, H. D. (2010). Segmentation of ultrasound breast images based on a neutrosophic method. *Optical Engineering*, *49*(11), 117001-117001.

Faculty of Computer Science and Mathematics
Universiti Malaysia Terengganu,
21030 Kuala Nerus, Terengganu, MALAYSIA
E-mail: sitinuraisyah.zainal@gmail.com, termimi@umt.edu.my[*]*, lazim_abdullah@umt.edu.my*

Department of Mathematics,
Wuhan University of Technology, Wuhan, China
430070, Wuhan, CHINA
E-mail: pengn@whut.edu.com

Received: 30 December 2024
Accepted: 17 June 2025

[*]Corresponding author