

COEFFICIENT ESTIMATES FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS INVOLVING THE EXTENDED q -FRACTIONAL OPERATOR

(Anggaran Pekali bagi Kelas Fungsi Analisis Tertentu Melibatkan Perluasan Pengoperasi Pecahan- q)

AREEJ ALOMAR, MASLINA DARUS* & KHALID M. K. ALSHAMMARI

ABSTRACT

In this paper, a new extended q -fractional operator using the convolution of the Al-Oboudi differential operator with the q -fractional integral operator $\mathcal{J}_q^{\alpha, \tau} f(z)$ is introduced. The q -fractional integral operator $\mathcal{J}_q^{\alpha, \tau} f(z)$ was recently defined and many new results are yet to be studied. From this extended operator, the coefficient problems involving the second Hankel determinant $H_2^2 f(z)$ and the Fekete-Szegő theorem are solved.

Keywords: analytic function; Hankel determinant; Fekete-Szegő theorem; quantum calculus.

ABSTRAK

Dalam kertas kerja ini, suatu perluasan pengoperasi pecahan- q baharu menggunakan konvolusi pengoperasi pembeza Al-Oboudi dengan pengoperasi kamiran pecahan- q $\mathcal{J}_q^{\alpha, \tau} f(z)$ diperkenalkan. Pengoperasi kamiran pecahan- q $\mathcal{J}_q^{\alpha, \tau} f(z)$ ini adalah yang terkini ditakrif dan banyak hasil baharu yang masih belum dikaji. Dari perluasan pengoperasi ini, anggaran pekali melibatkan penentu kedua Hankel $H_2^2 f(z)$ dan teorem Fekete-Szegő diselesaikan.

Kata kunci: fungsi analisis; penentu Hankel; teorem Fekete-Szegő, kalkulus kuantum.

1. Introduction

Let A denote the class of normalised analytic functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U}). \quad (1)$$

Further, we denote by S the subclass of A consisting of functions of the form Eq. (1), which are also univalent in $\mathbb{U}$.

In geometric function theory, q -calculus has been extensively used to deal with properties of certain classes defined in a specific domain. We provide here some essential definitions of q -calculus as given by Jackson (1909) and Aral *et al.* (2013), that is

$$\partial_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad (z \neq 0, z \in \mathbb{U}),$$

and

$$\int_0^z f(t) \partial_q t = z(1-q) \sum_{k=0}^{\infty} q^k f(zq^k), \quad q \in (0, 1).$$

Now when $k = 1, 2, 3, \dots$ and $z \in (0, 1)$, we can see that

$$\partial_q \left\{ \sum_{k=1}^{\infty} a_k z^k \right\} = \sum_{k=1}^{\infty} [k]_q a_k z^{k-1}$$

where

$$[k]_q = 1 + \sum_{m=1}^{k-1} q^m = \frac{1-q^k}{1-q}, \quad [0]_q = 0.$$

For any non-negative integer k , the q -number shift factorial is defined by:

$$[k]_q! = \begin{cases} 1 & k = 0 \\ [1]_q [2]_q [3]_q \cdots [k]_q & k = 1, 2, 3, \dots \end{cases}$$

In the past, many researchers studied generalised differential and integral operators and solved many interesting problems in univalent function theory. Here we will consider the well-known differential operator introduced by Al-Oboudi (2004) and the q -fractional integral operator given by Srivastava *et al.* (2023a). The definitions are given as follows:

Definition 1.1. (Al-Oboudi 2004) The Al-Oboudi differential operator is defined as follows:

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k, \quad 0 \leq \delta \leq 1, \quad n \in N_0 = N \cup \{0\}.$$

Definition 1.2. (Srivastava *et al.* 2023a) The q -fractional integral operator is defined as follows:

$$J_q^{\alpha, \tau} f(z) = z + \sum_{k=2}^{\infty} \frac{[k + \tau - 1]_q! \Gamma(k+1) \Gamma(\alpha+2)}{[\tau]_q! [k-1]_q! \Gamma(k+1+\alpha)} a_k z^k,$$

where $\tau \geq 0$, $\alpha > 0$, $\Gamma(k+1) = k! = k\Gamma(k)$, and $[k + \tau - 1]_q! = [k + \tau - 1]_q [k + \tau - 2]_q \cdots [1]_q$.

Now we introduce a new extended q -fractional operator defined by the convolution of the differential operator with the q -fractional integral operator, as given in Definition 1.1 and Definition 1.2, respectively. We get our new q -fractional operator as follows:

Definition 1.3. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belongs to the class A . Then

$$\begin{aligned} \mathcal{M}_{q, \delta}^{\alpha, \tau, n} f(z) &= J_q^{\alpha, \tau} D_\delta^n f(z) \\ &= J_q^{\alpha, \tau} D_\delta^n \left(z + \sum_{k=2}^{\infty} a_k z^k \right) \\ &= z + \sum_{k=2}^{\infty} \frac{[k + \tau - 1]_q! \Gamma(k+1) \Gamma(\alpha+2)}{[\tau]_q! [k-1]_q! \Gamma(k+1+\alpha)} [1 + (k-1)\delta]^n a_k z^k, \end{aligned} \quad (2)$$

where $\tau \geq 0$, $\alpha > 0$, $0 < q < 1$, $0 \leq \delta \leq 1$ and $n = 0, 1, 2, \dots$.

In this work, we are considering the problems of Hankel determinant and the Fekete-Szegő theorem. First, we will solve the Hankel determinant. Before that, we take a brief look at the Hankel determinant.

Noonan and Thomas (1976) introduced the function defined by Eq. (1) and provided the definition of the corresponding Hankel determinant as follows:

$$H_q(k) = \det \begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{pmatrix},$$

where the second Hankel determinant for $q = 2$ and $k = 2$ is defined by:

$$H_2(2) = \det \begin{pmatrix} a_2 & a_3 \\ a_3 & a_4 \end{pmatrix} = |a_2 a_4 - a_3^2|.$$

The study of this determinant has attracted significant attention from researchers such as Elhaddad and Darus (2020), Alshammari and Darus (2023), Srivastava *et al.* (2018), and Abubaker and Darus (2011). The sharp upper bounds for a second Hankel determinant of a class of analytic functions defined using the q -analogue of Ruscheweyh operator were studied by Aldweby and Darus (2014).

Thus, here we derive an upper bound for the second Hankel determinant of functions belonging to the new class $S(q, \tau, \delta, n)$ defined below.

Definition 1.4. Let f be given by Eq. (1). Then $f \in S(q, \tau, \delta, n)$ if it satisfies the inequality

$$\Re \left\{ \partial_q \left(\mathcal{M}_{q,\delta}^{\alpha,\tau,n} f(z) \right) \right\} > 0, \quad z \in \mathbb{U} \quad (3)$$

where $\mathcal{M}_{q,\delta}^{\alpha,\tau,n} f(z)$ is the new extension q -fractional operator defined in Eq. (2).

This class extends the family of functions with derivatives having a positive real, which was previously investigated by Macgregor (1962) and Janteng *et al.* (2006).

2. Preliminary Results

Let v be the family of all functions p analytic in $\mathbb{U}$ for which $\Re\{p(z)\} > 0$, and $p(z) = 1 + c_1 z + c_2 z^2 + \dots$, for $z \in \mathbb{U}$.

Lemma 2.1. (Pommerenke & Jensen 1975) If $p \in v$, then

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1^2|}{2}.$$

Lemma 2.2. (Duren 1983) If $p \in v$, then $|c_k| \leq 2$ for each $k \in N$ and the inequality is sharp.

Lemma 2.3. (Libera & Zlotkiewicz 1982) Let the function $p \in v$ be given by the power series. Then

$$c_2 = \frac{1}{2}(c_1^2 + x(4 - c_1^2))$$

for some x , $|x| \leq 1$, and

$$c_3 = \frac{1}{4}(c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z).$$

for some z , $|z| \leq 1$.

3. Main results

Our first result of Hankel determinant is contained in the following theorem.

Theorem 3.1. Let $f(z) \in S(q, \tau, \delta, n)$. Then

$$|a_2 a_4 - a_3^2| \leq \frac{9}{2[2]_q[4]_q \hat{h}_2 \hat{h}_4} + \frac{4}{[3]_q^2 \hat{h}_3^2},$$

where

$$\begin{aligned}\hat{h}_2 &= \frac{1}{[1]_q!} [\tau + 1]_q \frac{2!\Gamma(\alpha + 2)}{\Gamma(\alpha + 3)} [1 + \delta]^n, \\ \hat{h}_3 &= \frac{1}{[2]_q!} [\tau + 2]_q [\tau + 1]_q \frac{3!\Gamma(\alpha + 2)}{\Gamma(\alpha + 4)} [1 + 2\delta]^n,\end{aligned}$$

and

$$\hat{h}_4 = \frac{1}{[3]_q!} [\tau + 3]_q [\tau + 2]_q [\tau + 1]_q \frac{4!\Gamma(\alpha + 2)}{\Gamma(\alpha + 5)} [1 + 3\delta]^n.$$

Proof. Since $f(z) \in S(q, \tau, \delta, n)$. Then, from Eq. (3), there exists a function

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots \text{ in } v$$

such that

$$\partial_q(\mathcal{M}_{q,\delta}^{\alpha,\tau,n} f(z)) = p(z) \quad \text{for all } z \in \mathbb{U}.$$

This implies that

$$\partial_q \left(z + \hat{h}_2 a_2 z^2 + \hat{h}_3 a_3 z^3 + \hat{h}_4 a_4 z^4 + \dots \right) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots,$$

that is

$$\partial_q(z) + \hat{h}_2 a_2 \partial_q(z^2) + \hat{h}_3 a_3 \partial_q(z^3) + \hat{h}_4 a_4 \partial_q(z^4) + \dots = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots,$$

and yields

$$1 + [2]_q \hat{h}_2 a_2 z + [3]_q \hat{h}_3 a_3 z^2 + [4]_q \hat{h}_4 a_4 z^3 + \dots = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots.$$

Upon comparing the coefficients on both sides of this last equation, we are led to the following three equations:

$$a_2 = \frac{p_1}{[2]_q \hat{h}_2}, \tag{4}$$

$$a_3 = \frac{p_2}{[3]_q \hat{h}_3}, \tag{5}$$

$$a_4 = \frac{p_3}{[4]_q \hat{h}_4}. \tag{6}$$

Now, by applying Lemma 2.2 and Lemma 2.3 to Eqs. (5) and (6), we obtain

$$\begin{aligned}a_3 &= \frac{1}{[3]_q \hat{h}_3} \left(\frac{1}{2} (p_1^2 + x(4 - p_1^2)) \right) \\ &= \frac{1}{2[3]_q \hat{h}_3} (p_1^2 + x(4 - p_1^2))\end{aligned}$$

and

$$\begin{aligned} a_4 &= \frac{1}{[4]_q \hat{h}_4} \frac{1}{4} (p_1^3 + 2(4 - p_1^2)p_1x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z) \\ &= \frac{1}{4[4]_q \hat{h}_4} (p_1^3 + 2(4 - p_1^2)p_1x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z). \end{aligned}$$

Therefore,

$$\begin{aligned} |a_2a_4 - a_3^2| &= \left| \frac{p_1}{[2]_q \hat{h}_2} \left(\frac{1}{4[4]_q \hat{h}_4} (p_1^3 + 2(4 - p_1^2)p_1x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z) \right) \right. \\ &\quad \left. - \left(\frac{1}{2[3]_q \hat{h}_3} (p_1^2 + x(4 - p_1^2)) \right)^2 \right| \\ &= \left| \frac{1}{4[2]_q [4]_q \hat{h}_2 \hat{h}_4} \times \right. \\ &\quad \left\{ (p_1^4 + 2p_1^2(4 - p_1^2)x - p_1^2(4 - p_1^2)x^2 + 2p_1(4 - p_1^2)(1 - |x|^2)z) \right\} \\ &\quad \left. - \frac{1}{4[3]_q^2 \hat{h}_3^2} \{p_1^4 + 2p_1^2(4 - p_1^2)x + (4 - p_1^2)^2x^2\} \right|. \end{aligned}$$

Without loss of generality, we can assume that $p_1 = p$ and $p \in [0, 2]$, this implies that

$$|a_2a_4 - a_3^2| = \left| \frac{1}{4[2]_q [4]_q \hat{h}_2 \hat{h}_4} \left\{ p^4 + 2p^2(4 - p^2)x - p^2(4 - p^2)x^2 + 2p(4 - p^2)(1 - |x|^2)z \right\} \right. \\ \left. - \frac{1}{4[3]_q^2 \hat{h}_3^2} \left\{ p^4 + 2p^2(4 - p^2)x + (4 - p^2)^2x^2 \right\} \right|.$$

By using the triangle inequality, we find for $\zeta = |x|$; $|x| < 1$ that

$$\begin{aligned} |a_2a_4 - a_3^2| &\leq \frac{1}{4[2]_q [4]_q \hat{h}_2 \hat{h}_4} \{p^4 + 2p^2(4 - p^2)\zeta + p^2(4 - p^2)\zeta^2 + 2p(4 - p^2)(1 - \zeta^2)\} \\ &\quad + \frac{1}{4[3]_q^2 \hat{h}_3^2} \{p^4 + 2p^2(4 - p^2)\zeta + (4 - p^2)^2\zeta^2\}. \end{aligned}$$

If we denote the right-hand side of this last equation by $\mathbb{E}(p, \zeta)$, we obtain

$$\begin{aligned} \mathbb{E}(p, \zeta) &= \frac{1}{4[2]_q [4]_q \hat{h}_2 \hat{h}_4} \{p^4 + 2p^2(4 - p^2)\zeta + p^2(4 - p^2)\zeta^2 + 2p(4 - p^2)(1 - \zeta^2)\} \\ &\quad + \frac{1}{4[3]_q^2 \hat{h}_3^2} \{p^4 + 2p^2(4 - p^2)\zeta + (4 - p^2)^2\zeta^2\}; (p, \zeta) \in [0, 2] \times [0, 1]. \end{aligned}$$

We need to make use of the derivative test to maximize the function $\mathbb{E}(p, \zeta)$ in the rectangular region $[0, 2] \times [0, 1]$. Thus,

$$\frac{\partial \mathbb{E}}{\partial \zeta} = \frac{1}{4[2]_q [4]_q \hat{h}_2 \hat{h}_4} \{2p^2(4 - p^2) + 2p^2(4 - p^2)\zeta - 4p(4 - p^2)\zeta\}$$

$$+ \frac{1}{4[3]_q^2 \hat{h}_3^2} \{2p^2(4-p^2) + 2(4-p^2)^2 \zeta\} > 0.$$

Then we conclude that

$$\max_{0 \leq \zeta \leq 1} \mathbb{E}(p, \zeta) = \mathbb{E}(p, 1) = \Upsilon(p).$$

Therefore

$$\begin{aligned} \Upsilon(p) &= \frac{1}{4[2]_q[4]_q \hat{h}_2 \hat{h}_4} \{p^4 + 2p^2(4-p^2) + p^2(4-p^2)\} \\ &\quad + \frac{1}{4[3]_q^2 \hat{h}_3^2} \{p^4 + 2p^2(4-p^2) + (4-p^2)^2\}, \quad p \in [0, 2]. \\ &= \frac{1}{4[2]_q[4]_q \hat{h}_2 \hat{h}_4} \{p^4 + 3p^2(4-p^2)\} \\ &\quad + \frac{1}{4[3]_q^2 \hat{h}_3^2} \{p^4 + 2p^2(4-p^2) + (4-p^2)^2\}. \end{aligned}$$

Thus,

$$\begin{aligned} \Upsilon'(p) &= \frac{1}{4[2]_q[4]_q \hat{h}_2 \hat{h}_4} \{4p^3 + 6p(4-p^2) - 6p^3\} \\ &\quad + \frac{1}{4[3]_q^2 \hat{h}_3^2} \{4p^3 + 4p(4-p^2) - 4p^3 - 4p(4-p^2)\} \\ &= \frac{1}{4[2]_q[4]_q \hat{h}_2 \hat{h}_4} \{6p(4-p^2) - 2p^3\} \\ &= \frac{1}{4[2]_q[4]_q \hat{h}_2 \hat{h}_4} \{24p - 6p^3 - 2p^3\} \\ &= \frac{1}{4[2]_q[4]_q \hat{h}_2 \hat{h}_4} \{24p - 8p^3\} \\ &= \frac{1}{4[2]_q[4]_q \hat{h}_2 \hat{h}_4} 8p\{3 - p^2\}. \end{aligned}$$

However, the function $\Upsilon(p)$ increases within the interval $[0, \sqrt{3}]$ and decreases within the interval $[\sqrt{3}, 2]$. Therefore, we conclude that $\Upsilon(p)$ has its maximum value in the interval $[0, 2]$ when $p = \sqrt{3}$. We can eventually have

$$\begin{aligned} |a_2 a_4 - a_3^2| &\leq \mathbb{E}(p, \zeta) \\ &\leq \max_{0 \leq \zeta \leq 1, 0 \leq p \leq 2} \mathbb{E}(p, \zeta) \\ &= \mathbb{E}(\sqrt{3}, 1) \end{aligned}$$

$$= \frac{1}{4[2]_q[4]_q \hat{h}_2 \hat{h}_4} \left\{ (\sqrt{3})^4 + 2(\sqrt{3})^2(4 - (\sqrt{3})^2) + (\sqrt{3})^2(4 - (\sqrt{3})^2) \right\}$$

$$\begin{aligned}
 & + \frac{1}{4[3]_q^2 \hat{h}_3^2} \left\{ (\sqrt{3})^4 + 2(\sqrt{3})^2(4 - (\sqrt{3})^2) + (4 - (\sqrt{3})^2)^2 \right\} \\
 & = \frac{1}{4[2]_q[4]_q \hat{h}_2 \hat{h}_4} \{9 + 6(1) + 3(1)\} + \frac{1}{4[3]_q^2 \hat{h}_3^2} \{9 + 6(1) + (1)^2\} \\
 & = \frac{18}{4[2]_q[4]_q \hat{h}_2 \hat{h}_4} + \frac{16}{4[3]_q^2 \hat{h}_3^2} \\
 & = \frac{9}{2[2]_q[4]_q \hat{h}_2 \hat{h}_4} + \frac{4}{[3]_q^2 \hat{h}_3^2}.
 \end{aligned}$$

Hence, the result is obtained. $\square$

Next, we give our second result that is on the Fekete-Szegő theorem. It is well-known that the Hankel determinant, when $q = 2$, $k = 1$ and $a_1 = 1$, we have the Fekete-Szegő functional:

$$H_2(1) = \det \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} = |a_1 a_3 - a_2^2|.$$

This problem has been extensively researched and is still a significant and active study area in classical complex geometry. For an overview of its history, see the papers by Issa and Darus (2023), Srivastava *et al.* (2023b), Darus and Thomas (2000), and Cho and Owa (2003). We now define the following class to get our result.

Definition 3.2. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $a_k \geq 0$, $\in \mathbb{TR}_{\lambda, q, \delta}^{\alpha, \tau, n}(\beta, \gamma)$ where, $0 \leq \lambda \leq 1$, $0 < q < 1$, $0 \leq \delta \leq 1$, $\alpha > 0$, $\tau \geq 0$, $\beta > 0$, $0 < \gamma \leq 1$, and $n = 0, 1, 2, \dots$. If there exists a function $\phi \in A$, then

$$\left| \arg \left((1 - \lambda) \frac{\mathcal{M}_{q, \delta}^{\alpha, \tau, n} f(z)}{\phi(z)} + \lambda \frac{z \partial_q \left(\mathcal{M}_{q, \delta}^{\alpha, \tau, n} f(z) \right)}{\phi(z)} \right) \right| < \frac{\pi \beta}{2}, \quad z \in \mathbb{U}, \quad (7)$$

where, $\mathcal{M}_{q, \delta}^{\alpha, \tau, n} f(z)$ is the generalised q -fractional operator that we introduced in Definition 1.3, and $\phi(z) = z + b_2 z^2 + b_3 z^3 + \dots$ is analytic and satisfies the requirements

$$\left| \arg \left(\frac{\phi(z) * \psi(z)}{\phi(z) * \eta(z)} \right) \right| < \frac{\pi \gamma}{2}, \quad (8)$$

for the functions $\psi(z) = z + \sum_{k=2}^{\infty} \sigma_k z^k$ and $\eta(z) = z + \sum_{k=2}^{\infty} \zeta_k z^k$, both are analytic in $\mathbb{U}$ and $\phi(z) * \eta(z) \neq 0$, $\sigma_k \geq 0$, $\zeta_k \geq 0$, and $\sigma_k > \zeta_k$ ($k \geq 2$).

Note that, by substituting $m = \tau = 0$, with $q \rightarrow 1-$ into the Definition 3.2, the class $\mathbb{TR}_{\lambda, q, \delta}^{\alpha, \tau, n}(\beta, \gamma)$ represents the set of strongly α -close-to-convex with order equal β defined by Darus (2005).

The Fekete-Szegő problem for the class given in Definition 3.2 is given as follows:

Theorem 3.3. Let $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$, be a function in the class $\mathbb{TR}_{\lambda, q, \delta}^{\alpha, \tau, n}(\beta, \gamma)$ where $0 \leq \lambda \leq 1$, $0 < q < 1$, $0 \leq \delta \leq 1$, $\alpha > 0$, $\tau \geq 0$, $\beta > 0$, $0 < \gamma \leq 1$, and $n = 0, 1, 2, \dots$. If $\frac{U_2^2}{\beta U_3} < \mu$, and

$$\mu > \frac{U_2^4 J_3^4 [(\epsilon + 2\zeta_2) + \nu]}{J_4 U_3 [\mu \nu U_3 J_4 - 2\nu U_2^2 J_3^2 - 2\mu \nu U_3 J_4 J_3^2 + (\epsilon + 2\zeta_2) + \nu U_2^2 J_3^4]},$$

where $\mu(\nu U_3 J_4 - 2\nu U_3 J_4 J_3^2) - [2\nu U_2^2 J_3^2 - (\epsilon + 2\zeta_2) - \nu U_2^2 J_3^4] > 0$, $U_2 = 1 + ([2]_q - 1)\lambda$, $U_3 = 1 + ([3]_q - 1)\lambda$, $\nu = \sigma_3 - \zeta_3$ and $\epsilon = \sigma_2 - \zeta_2$. Then the sharp inequality is obtained as follows

$$|a_3 - \mu a_2^2| \leq \frac{4\beta J_4 (1 + ([3]_q - 1)\lambda) (\epsilon\beta + 2\gamma) \mu - 2\beta (1 + ([2]_q - 1)\lambda)^2 (\beta\epsilon J_3^2 + 2\gamma)}{J_3^2 J_4 \epsilon (1 + ([2]_q - 1)\lambda)^2 (1 + ([3]_q - 1)\lambda)} + \frac{2\gamma^2 \left\{ 2\mu J_4 (1 + ([3]_q - 1)\lambda) \nu - J_3^2 \epsilon (\epsilon + 2\zeta_2) (1 + ([2]_q - 1)\lambda)^2 \right\}}{J_4 \nu \epsilon^2 (1 + ([2]_q - 1)\lambda)^2 (1 + ([3]_q - 1)\lambda)},$$

where $J_3 = [\tau + 1]_q \frac{2!\Gamma(\alpha+2)}{\Gamma(\alpha+3)} [1 + \delta]^n$ and $J_4 = \frac{1}{[2]_q!} [\tau + 2]_q [\tau + 1]_q \frac{3!\Gamma(\alpha+2)}{\Gamma(\alpha+4)} [1 + 2\delta]^n$.

Proof. Since $f(z)$ belongs to $\mathbb{TR}_{\lambda,q,\delta}^{\alpha,\tau,n}(\beta, \gamma)$, from Eq. (7), there is $p(z) = 1 + \rho_1 z + \rho_2 z^2 + \dots$, where $\rho(z) \in v$ and $\phi(z) = z + b_2 z^2 + b_3 z^3 + \dots \in A$ such that

$$(1 - \lambda) \mathcal{M}_{q,\delta}^{\alpha,\tau,n} f(z) + \lambda z \partial_q \left(\mathcal{M}_{q,\delta}^{\alpha,\tau,n} f(z) \right) = \phi(z) \rho^\beta(z),$$

which imply

$$\begin{aligned} & (1 - \lambda) \{ z + a_2 J_3 z^2 + a_3 J_4 z^3 + \dots \} \\ & + \lambda z \{ 1 + [2]_q a_2 J_3 z + [3]_q a_3 J_4 z^2 + \dots \} \\ & = (z + b_2 z^2 + b_3 z^3 + \dots) (1 + \rho_1 z + \rho_2 z^2 + \rho_3 z^3 + \dots)^\beta. \end{aligned}$$

By expanding the given expression, we obtain

$$(1 + \rho_1 z + \rho_2 z^2 + \rho_3 z^3 + \dots)^\beta = 1 + \beta \rho_1 z + \left(\beta \rho_2 + \left(\frac{\beta^2}{2} - \frac{\beta}{2} \right) \rho_1^2 \right) z^2 + \dots.$$

Therefore,

$$\begin{aligned} & \{ (1 - \lambda) + \lambda \} z + \{ (1 - \lambda) J_3 + [2]_q \lambda J_3 \} a_2 z^2 + \\ & \{ (1 - \lambda) J_4 + [3]_q \lambda J_4 \} a_3 z^3 + \dots \\ & = z + \{ \rho_1 \beta + b_2 \} z^2 + \left\{ \rho_1 \beta b_2 + \beta \rho_2 + \left(\frac{\beta^2}{2} - \frac{\beta}{2} \right) \rho_1^2 + b_3 \right\} z^3 + \dots. \end{aligned}$$

By matching the coefficients in the final equation, we may deduce

$$\{ 1 + \lambda([2]_q - 1) \} J_3 a_2 = \rho_1 \beta + b_2, \quad (9)$$

and

$$\{ 1 + \lambda([3]_q - 1) \} J_4 a_3 = \rho_1 \beta b_2 + \beta \rho_2 + \left(\frac{\beta^2}{2} - \frac{\beta}{2} \right) \rho_1^2 + b_3. \quad (10)$$

Furthermore, from Eq. (8), we may conclude that there is a function in v , denoted as $\xi(z) = 1 + \xi_1 z + \xi_2 z^2 + \xi_3 z^3 + \dots$, such that

$$\phi(z) * \psi(z) = (\phi(z) * \eta(z)) \xi^\gamma(z),$$

and

$$\begin{aligned} & (z + b_2 z^2 + b_3 z^3 + \dots) * (z + \sigma_2 z^2 + \sigma_3 z^3 + \dots) \\ &= \{(z + b_2 z^2 + b_3 z^3 + \dots) * (z + \zeta_2 z^2 + \zeta_3 z^3 + \dots)\} (1 + \xi_1 z + \xi_2 z^2 + \xi_3 z^3 + \dots)^\gamma. \end{aligned}$$

Therefore,

$$\begin{aligned} & z + b_2 \sigma_2 z^2 + b_3 \sigma_3 z^3 + \dots = \{z + b_2 \zeta_2 z^2 + b_3 \zeta_3 z^3 + \dots\} \\ & \left\{ 1 + \gamma \xi_1 z + \left(\gamma \xi_2 + \left(\frac{\gamma^2}{2} - \frac{\gamma}{2} \right) \xi_1^2 \right) z^2 + \dots \right\} \\ &= z + (\xi_1 \gamma + \zeta_2 b_2) z^2 + \left(\xi_1 \gamma b_2 \zeta_2 + \gamma \xi_2 + \left(\frac{\gamma^2}{2} - \frac{\gamma}{2} \right) \xi_1^2 + \zeta_3 b_3 \right) z^3 + \dots. \end{aligned}$$

By equating the coefficients in the last expression yields

$$b_2 \sigma_2 = \xi_1 \gamma + \zeta_2 b_2.$$

Hence,

$$b_2 = \frac{\xi_1 \gamma}{\epsilon}, \epsilon = \sigma_2 - \zeta_2 > 0. \quad (11)$$

Furthermore,

$$\begin{aligned} b_3 \sigma_3 &= \xi_1 \gamma b_2 \zeta_2 + \gamma \xi_2 + \left(\frac{\gamma^2}{2} - \frac{\gamma}{2} \right) \xi_1^2 + \zeta_3 b_3 \\ b_3 \sigma_3 - \zeta_3 b_3 &= \xi_1 \gamma b_2 \zeta_2 + \gamma \xi_2 + \frac{\gamma^2}{2} \xi_1^2 - \frac{\gamma}{2} \xi_1^2 \\ (\sigma_3 - \zeta_3) b_3 &= \gamma \left(\xi_1 b_2 \zeta_2 + \xi_2 + \frac{\gamma}{2} \xi_1^2 - \frac{1}{2} \xi_1^2 \right) \\ \nu b_3 &= \gamma \left(\xi_1 \frac{\xi_1 \gamma}{\epsilon} \zeta_2 + \xi_2 + \frac{\gamma}{2} \xi_1^2 - \frac{1}{2} \xi_1^2 \right) \\ &= \gamma \left(\xi_2 + \xi_1^2 \frac{\gamma}{\epsilon} \zeta_2 + \frac{\gamma}{2} \xi_1^2 - \frac{1}{2} \xi_1^2 \right) \\ &= \gamma \left(\xi_2 + \xi_1^2 \left(\frac{\gamma}{\epsilon} \zeta_2 + \frac{\gamma}{2} - \frac{1}{2} \right) \right) \\ &= \gamma \left(\xi_2 + \xi_1^2 \left(\frac{2\gamma \zeta_2}{2\epsilon} + \frac{\epsilon \gamma}{2\epsilon} - \frac{\epsilon}{2\epsilon} \right) \right) \end{aligned}$$

This implies that

$$b_3 = \frac{\gamma}{\nu} \left\{ \xi_2 + \left(\frac{\gamma(\epsilon + 2\zeta_2) - \epsilon}{2\epsilon} \right) \xi_1^2 \right\}, \nu = \sigma_3 - \zeta_3. \quad (12)$$

From Eqs. (9)-(12), we get

$$a_2 = \frac{1}{\{1 + \lambda([2]_q - 1)\} J_3} (\rho_1 \beta + \frac{\xi_1 \gamma}{\epsilon}) \quad (13)$$

and

$$a_3 = \frac{1}{\{1 + \lambda([3]_q - 1)\} J_4} \times \left\{ \frac{\xi_1 \rho_1 \beta \gamma}{\epsilon} + \beta \rho_2 + \left(\frac{\beta^2}{2} - \frac{\beta}{2} \right) \rho_1^2 + \frac{\gamma}{\nu} \left(\xi_2 + \left(\frac{\gamma(\epsilon + 2\zeta_2) - \epsilon}{2\epsilon} \right) \xi_1^2 \right) \right\}. \quad (14)$$

Using Eqs. (9)-(14), we obtain the required result as shown below:

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{\beta \gamma \xi_1 \rho_1}{\epsilon \{1 + \lambda([3]_q - 1)\} J_4} + \frac{\beta \rho_2}{\{1 + \lambda([3]_q - 1)\} J_4} + \frac{\beta^2 \rho_1^2}{2 \{1 + \lambda([3]_q - 1)\} J_4} \\ &\quad - \frac{\beta \rho_1^2}{2 \{1 + \lambda([3]_q - 1)\} J_4} + \frac{\gamma \xi_2}{\nu \{1 + \lambda([3]_q - 1)\} J_4} + \frac{\gamma^2 (\epsilon + 2\zeta_2) \xi_1^2}{2\epsilon \nu \{1 + \lambda([3]_q - 1)\} J_4} \\ &\quad - \frac{\gamma \epsilon \xi_1^2}{2\epsilon \nu \{1 + \lambda([3]_q - 1)\} J_4} - \frac{\mu \rho_1^2 \beta^2}{\{1 + \lambda([2]_q - 1)\}^2 J_3^2} - \frac{2\mu \gamma \beta \xi_1 \rho_1}{\epsilon \{1 + \lambda([2]_q - 1)\}^2 J_3^2} \\ &\quad - \frac{\mu \gamma^2 \xi_1^2}{\epsilon^2 \{1 + \lambda([2]_q - 1)\}^2 J_3^2} \\ &= \left\{ \frac{\beta \rho_2}{\{1 + \lambda([3]_q - 1)\} J_4} - \frac{\beta \rho_1^2}{2 \{1 + \lambda([3]_q - 1)\} J_4} \right\} \\ &\quad + \left\{ \frac{\gamma \xi_2}{\nu \{1 + \lambda([3]_q - 1)\} J_4} - \frac{\gamma \epsilon \xi_1^2}{2\epsilon \nu \{1 + \lambda([3]_q - 1)\} J_4} \right\} \\ &\quad + \left\{ \frac{\beta^2 \rho_1^2}{2 \{1 + \lambda([3]_q - 1)\} J_4} - \frac{\mu \rho_1^2 \beta^2}{\{1 + \lambda([2]_q - 1)\}^2 J_3^2} \right\} \\ &\quad + \left\{ \frac{\beta \gamma \xi_1 \rho_1}{\epsilon \{1 + \lambda([3]_q - 1)\} J_4} - \frac{2\mu \gamma \beta \xi_1 \rho_1}{\epsilon \{1 + \lambda([2]_q - 1)\}^2 J_3^2} \right\} \\ &\quad + \left\{ \frac{\gamma^2 (\epsilon + 2\zeta_2) \xi_1^2}{2\epsilon \nu \{1 + \lambda([3]_q - 1)\} J_4} - \frac{\mu \gamma^2 \xi_1^2}{\epsilon^2 \{1 + \lambda([2]_q - 1)\}^2 J_3^2} \right\}, \end{aligned}$$

that gives

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{\beta}{\{1 + \lambda([3]_q - 1)\} J_4} \left\{ \rho_2 - \frac{\rho_1^2}{2} \right\} + \frac{\gamma}{\nu \{1 + \lambda([3]_q - 1)\} J_4} \left\{ \xi_2 - \frac{\xi_1^2}{2} \right\} \\ &\quad + \beta^2 \rho_1^2 \left\{ \frac{\{1 + \lambda([2]_q - 1)\}^2 J_3^2 - 2\mu \{1 + \lambda([3]_q - 1)\} J_4}{2 \{1 + \lambda([3]_q - 1)\} \{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} \\ &\quad + \beta \gamma \xi_1 \rho_1 \left\{ \frac{\{1 + \lambda([2]_q - 1)\}^2 J_3^2 - 2\mu \{1 + \lambda([3]_q - 1)\} J_4}{\epsilon \{1 + \lambda([3]_q - 1)\} \{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} \\ &\quad + \gamma^2 \xi_1^2 \left\{ \frac{\epsilon(\epsilon + 2\zeta_2) \{1 + \lambda([2]_q - 1)\}^2 J_3^2 - 2\mu \nu \{1 + \lambda([3]_q - 1)\} J_4}{2\nu \epsilon^2 \{1 + \lambda([3]_q - 1)\} \{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\}. \quad (15) \end{aligned}$$

Here, we establish the proof using the approach suggested by London (1993). Assuming that $a_3 - \mu a_2^2$ remains positive, we use equation Eq. (15) and substitute $\xi_1 = 2re^{i\theta}$ and $\rho_1 = 2Re^{i\phi}$, $0 \leq r \leq 1$, $0 \leq R \leq 1$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq 2\pi$. Consequently, we obtain at the

following result:

$$\begin{aligned} \operatorname{Re}(a_3 - \mu a_2^2) &= \frac{\beta}{\{1 + \lambda([3]_q - 1)\}J_4} \operatorname{Re} \left\{ \rho_2 - \frac{\rho_1^2}{2} \right\} + \frac{\gamma}{\nu\{1 + \lambda([3]_q - 1)\}J_4} \operatorname{Re} \left\{ \xi_2 - \frac{\xi_1^2}{2} \right\} \\ &+ \beta^2 \left\{ \frac{\{1 + \lambda([2]_q - 1)\}^2 J_3^2 - 2\mu\{1 + \lambda([3]_q - 1)\}J_4}{2\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} \operatorname{Re} \rho_1^2 \\ &+ \beta\gamma \left\{ \frac{\{1 + \lambda([2]_q - 1)\}^2 J_3^2 - 2\mu\{1 + \lambda([3]_q - 1)\}J_4}{\epsilon\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} \operatorname{Re} \xi_1 \rho_1 \\ &+ \gamma^2 \left\{ \frac{\epsilon(\epsilon + 2\zeta_2)\{1 + \lambda([2]_q - 1)\}^2 J_3^2 - 2\mu\nu\{1 + \lambda([3]_q - 1)\}J_4}{2\nu\epsilon^2\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} \operatorname{Re} \xi_1^2. \end{aligned}$$

Thus by referring to Lemma 2.1, we have

$$\begin{aligned} \operatorname{Re}(a_3 - \mu a_2^2) &= \frac{\beta}{\{1 + \lambda([3]_q - 1)\}J_4} \left\{ 2 - \frac{|\rho_1|^2}{2} \right\} + \frac{\gamma}{\nu\{1 + \lambda([3]_q - 1)\}J_4} \left\{ 2 - \frac{|\xi_1|^2}{2} \right\} \\ &+ \beta^2 \left\{ \frac{\{1 + \lambda([2]_q - 1)\}^2 J_3^2 - 2\mu\{1 + \lambda([3]_q - 1)\}J_4}{2\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} \operatorname{Re} \rho_1^2 \\ &+ \beta\gamma \left\{ \frac{\{1 + \lambda([2]_q - 1)\}^2 J_3^2 - 2\mu\{1 + \lambda([3]_q - 1)\}J_4}{\epsilon\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} \operatorname{Re} \xi_1 \rho_1 \\ &+ \gamma^2 \left\{ \frac{\epsilon(\epsilon + 2\zeta_2)\{1 + \lambda([2]_q - 1)\}^2 J_3^2 - 2\mu\nu\{1 + \lambda([3]_q - 1)\}J_4}{2\nu\epsilon^2\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} \operatorname{Re} \xi_1^2. \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{Re}(a_3 - \mu a_2^2) &\leq \frac{2\beta}{\{1 + \lambda([3]_q - 1)\}J_4} (1 - R^2) + \frac{2\gamma}{\nu\{1 + \lambda([3]_q - 1)\}J_4} (1 - r^2) \\ &+ 2\beta^2 \left\{ \frac{\{1 + \lambda([2]_q - 1)\}^2 J_3^2 - 2\mu\{1 + \lambda([3]_q - 1)\}J_4}{\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} R^2 \cos 2\theta \\ &+ 4\beta\gamma \left\{ \frac{\{1 + \lambda([2]_q - 1)\}^2 J_3^2 - 2\mu\{1 + \lambda([3]_q - 1)\}J_4}{\epsilon\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} rR \cos(\phi + \theta) \\ &+ 2\gamma^2 \left\{ \frac{\epsilon(\epsilon + 2\zeta_2)\{1 + \lambda([2]_q - 1)\}^2 J_3^2 - 2\mu\nu\{1 + \lambda([3]_q - 1)\}J_4}{\nu\epsilon^2\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} r^2 \cos 2\phi \\ &\leq \frac{2\beta}{\{1 + \lambda([3]_q - 1)\}J_4} (1 - R^2) + \frac{2\gamma}{\nu\{1 + \lambda([3]_q - 1)\}J_4} (1 - r^2) \\ &+ 2\beta^2 \left\{ \frac{2\mu\{1 + \lambda([3]_q - 1)\}J_4 - \{1 + \lambda([2]_q - 1)\}^2 J_3^2}{\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} R^2 \\ &+ 4\beta\gamma \left\{ \frac{2\mu\{1 + \lambda([3]_q - 1)\}J_4 - \{1 + \lambda([2]_q - 1)\}^2 J_3^2}{\epsilon\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} rR \\ &+ 2\gamma^2 \left\{ \frac{2\mu\nu\{1 + \lambda([3]_q - 1)\}J_4 - \epsilon(\epsilon + 2\zeta_2)\{1 + \lambda([2]_q - 1)\}^2 J_3^2}{\nu\epsilon^2\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} r^2 = H(r, R). \end{aligned}$$

Letting λ , β , and μ be fixed, and computing the partial derivatives of $H(r, R)$ with respect to both r and R when $\lambda \leq 1$, $\beta \geq 1$ and $\mu \geq 1$, we determine the sign of $H_{rr}H_{RR} - (H_{rR})^2$,

as shown below:

$$\begin{aligned}
 H_{rr}H_{RR} - (H_{rR})^2 = & \left\{ \frac{-4\gamma}{\nu\{1 + \lambda([3]_q - 1)\}J_4} + 4\gamma^2 \left(\frac{2\mu\nu\{1 + \lambda([3]_q - 1)\}J_4 - \epsilon(\epsilon + 2\zeta_2)\{1 + \lambda([2]_q - 1)\}^2 J_3^2}{\nu\epsilon^2\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right) \right\} \\
 & \times \left\{ \frac{-4\beta}{\{1 + \lambda([3]_q - 1)\}J_4} + 4\beta^2 \left(\frac{2\mu\{1 + \lambda([3]_q - 1)\}J_4 - \{1 + \lambda([2]_q - 1)\}^2 J_3^2}{\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right) \right\} \\
 & - 16\beta^2\gamma^2 \left\{ \frac{2\mu\{1 + \lambda([3]_q - 1)\}J_4 - \{1 + \lambda([2]_q - 1)\}^2 J_3^2}{\epsilon\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\}^2,
 \end{aligned}$$

this implies

$$\begin{aligned}
 H_{rr}H_{RR} - (H_{rR})^2 = & \frac{16\gamma\beta}{\nu\{1 + \lambda([3]_q - 1)\}^2 J_4^2} \\
 & - \frac{16\gamma\beta^2\{2\mu\{1 + \lambda([3]_q - 1)\}J_4 - \{1 + \lambda([2]_q - 1)\}^2 J_3^2\}}{\nu\{1 + \lambda([3]_q - 1)\}^2\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4^2} \\
 & - \frac{16\gamma^2\beta\{2\mu\nu\{1 + \lambda([3]_q - 1)\}J_4 - \epsilon(\epsilon + 2\zeta_2)\{1 + \lambda([2]_q - 1)\}^2 J_3^2\}}{\nu\epsilon^2\{1 + \lambda([3]_q - 1)\}^2\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4^2} \\
 & 16\gamma^2\beta^2\{2\mu\nu\{1 + \lambda([3]_q - 1)\}J_4 - \epsilon(\epsilon + 2\zeta_2)\{1 + \lambda([2]_q - 1)\}^2 J_3^2\} \\
 & + \frac{\{2\mu\{1 + \lambda([3]_q - 1)\}J_4 - \{1 + \lambda([2]_q - 1)\}^2 J_3^2\}}{\nu\epsilon^2\{1 + \lambda([3]_q - 1)\}^2\{1 + \lambda([2]_q - 1)\}^4 J_3^2 J_4^2} \\
 & - 16\beta^2\gamma^2 \left\{ \frac{2\mu\{1 + \lambda([3]_q - 1)\}J_4 - \{1 + \lambda([2]_q - 1)\}^2 J_3^2}{\epsilon\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\}^2 \\
 & < 0.
 \end{aligned}$$

As a result, the function $H(r, R)$ reaches its maximum value at the boundary of the unit disk. Therefore, our inequality comes from the observation that

$$\begin{aligned}
 H(r, R) & \leq H(1, 1) \\
 & = 2\beta^2 \left\{ \frac{2\mu\{1 + \lambda([3]_q - 1)\}J_4 - \{1 + \lambda([2]_q - 1)\}^2 J_3^2}{\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} \\
 & + 4\beta\gamma \left\{ \frac{2\mu\{1 + \lambda([3]_q - 1)\}J_4 - \{1 + \lambda([2]_q - 1)\}^2 J_3^2}{\epsilon\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} \\
 & + 2\gamma^2 \left\{ \frac{2\mu\nu\{1 + \lambda([3]_q - 1)\}J_4 - \epsilon(\epsilon + 2\zeta_2)\{1 + \lambda([2]_q - 1)\}^2 J_3^2}{\nu\epsilon^2\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\} \\
 & 2\beta^2\epsilon\{2\mu\{1 + \lambda([3]_q - 1)\}J_4 - \{1 + \lambda([2]_q - 1)\}^2 J_3^2\} \\
 & = \frac{+4\beta\gamma\{2\mu\{1 + \lambda([3]_q - 1)\}J_4 - \{1 + \lambda([2]_q - 1)\}^2 J_3^2\}}{\epsilon\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \\
 & + 2\gamma^2 \left\{ \frac{2\mu\nu\{1 + \lambda([3]_q - 1)\}J_4 - \epsilon(\epsilon + 2\zeta_2)\{1 + \lambda([2]_q - 1)\}^2 J_3^2}{\nu\epsilon^2\{1 + \lambda([3]_q - 1)\}\{1 + \lambda([2]_q - 1)\}^2 J_3^2 J_4} \right\}
 \end{aligned}$$

$$\begin{aligned}
& 4\beta^2\epsilon\mu\{1+\lambda([3]_q-1)\}J_4-2\beta^2\epsilon\{1+\lambda([2]_q-1)\}^2J_3^2 \\
&= \frac{+8\beta\gamma\mu\{1+\lambda([3]_q-1)\}J_4-4\beta\gamma\{1+\lambda([2]_q-1)\}^2J_3^2}{\epsilon\{1+\lambda([3]_q-1)\}\{1+\lambda([2]_q-1)\}^2J_3^2J_4} \\
&+ 2\gamma^2\left\{\frac{2\mu\nu\{1+\lambda([3]_q-1)\}J_4-\epsilon(\epsilon+2\zeta_2)\{1+\lambda([2]_q-1)\}^2J_3^2}{\nu\epsilon^2\{1+\lambda([3]_q-1)\}\{1+\lambda([2]_q-1)\}^2J_3^2J_4}\right\} \\
&= \frac{4\beta\mu J_4\{1+\lambda([3]_q-1)\}(\epsilon\beta+2\gamma)-2\beta\{1+\lambda([2]_q-1)\}^2(\beta\epsilon J_3^2+2\gamma)}{\epsilon\{1+\lambda([3]_q-1)\}\{1+\lambda([2]_q-1)\}^2J_3^2J_4} \\
&+ 2\gamma^2\left\{\frac{2\mu\nu\{1+\lambda([3]_q-1)\}J_4-\epsilon(\epsilon+2\zeta_2)\{1+\lambda([2]_q-1)\}^2J_3^2}{\nu\epsilon^2\{1+\lambda([3]_q-1)\}\{1+\lambda([2]_q-1)\}^2J_3^2J_4}\right\}.
\end{aligned}$$

By letting $q \rightarrow 1-$ and setting $n = \tau = 0$, $\xi_1 = \rho_1 = 2i$ and $\xi_2 = \rho_2 = -2$ in Eq. (15), we obtain the equality. $\square$

4. Conclusion

In this study, we introduced a new operator that extends the classical framework of the univalent function theory. Through this operator, we have successfully generalised the upper bounds of the Hankel determinant. In addition, we have expanded the Fekete-Szegő inequalities. For future work, we may explore the implications of our results in special functions and investigate the possibility of identifying new properties related to the extended q -fractional operator.

References

- Abubaker A. & Darus M. 2011. Hankel determinant for a class of analytic functions involving a generalized linear differential operator. *International Journal of Pure and Applied Mathematics* **69**(4): 429–435.
- Al-Oboudi F.M. 2004. On univalent functions defined by a generalized Sălăgean operator. *International Journal of Mathematics and Mathematical sciences* **27**: 1429–1436.
- Aldweby H. & Darus M. 2014. Some subordination results on q -analogue of Ruscheweyh differential operator. *Abstract and Applied Analysis* **2014**: 958563.
- Alshammari K. & Darus M. 2023. Hankel determinant for a class of analytic functions involving the q -Ruscheweyh derivative and the symmetric differential operator. *International Journal of Mathematics and Computer Science* **18**(1): 47–53.
- Aral A., Gupta V. & Agarwal R P. 2013. *Applications of q -Calculus in Operator Theory*. New York: Springer.
- Cho N.E. & Owa S. 2003. On the Fekete-Szegő problem for strongly α -quasiconvex functions. *Tamkang Journal of Mathematics* **34**(1): 21–28.
- Darus M. 2005. Fekete-Szegő problem for α -close-to-convex functions. *Journal of Computational Mathematics and Optimization (JOCMO)* **1**(1): 45–53.
- Darus M. & Thomas D.K. 2000. The Fekete-Szegő problem for strongly close-to-convex functions. *Scientiae Mathematicae* **3**(2): 201–212.
- Duren P.L. 1983. *Univalent Functions*. New York: Springer-Verlag.
- Elhaddad S. & Darus M. 2020. Second Hankel determinant for a subclass of analytic functions involving the q -analogue of Ruscheweyh operator. *Journal of Quality Measurement and Analysis* **16**(1): 99–106.
- Issa A.S. & Darus M. 2023. Fekete-Szegő problem of strongly α -close-to-convex functions associated with generalized fractional operator. *Georgian Mathematical Journal* **30**(1): 111–116.
- Jackson F.H. 1909. On q -functions and a certain difference operator. *Transactions of the Royal Society of Edinburgh* **46**(2): 253–281.
- Janteng A., Halim S.A. & Darus M. 2006. Coefficient inequality for a function whose derivative has a positive real part. *Journal of Inequalities in Pure and Applied Mathematics* **7**(2): 1–5.

- Libera R.J. & Zlotkiewicz E.J. 1982. Early coefficients of the inverse of a regular convex function. *Proceedings of the American Mathematical Society* **85**(2): 225–230.
- London R.R. 1993. Fekete-Szegő inequalities for close-to-convex functions. *Proceedings of the American Mathematical Society* **117**(4): 947–950.
- Macgregor T.H. 1962. Functions whose derivative has a positive real part. *Transactions of the American Mathematical Society* **104**(3): 532–537.
- Noonan J.W. & Thomas D.K. 1976. On the second Hankel determinant of a really mean p -valent functions. *Transactions of the American Mathematical Society* **223**: 337–346.
- Pommerenke C. & Jensen G. 1975. *Univalent Functions: With a Chapter on Quadratic Differentials*. Göttingen: Vandenhoeck & Ruprecht.
- Srivastava H.M., Alshammari K. & Darus M. 2023a. A new q -fractional integral operator and its applications to the coefficient problem involving the second Hankel determinant for q -starlike and q -convex functions. *Journal of Nonlinear and Variational Analysis* **7**(6): 985–994.
- Srivastava H.M., Altinkaya Ş. & Yalcin S. 2018. Hankel determinant for a subclass of bi-univalent functions defined by using a symmetric q -derivative operator. *Filomat* **32**(2): 503–516.
- Srivastava H.M., Shaba T.G., Murugusundaramoorthy G., Wanas A.K. & Oros G.I. 2023b. The Fekete-Szegő functional and the Hankel determinant for a certain class of analytic functions involving the Hohlov operator. *AIMS Mathematics* **8**(1): 340–360.

Department of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 UKM Bangi
Selangor, MALAYSIA.
*E-mail: p142924@siswa.ukm.edu.my, maslina@ukm.edu.my**

Department of Mathematics
University of Ha'il, College of Science
Ha'il, 55425
SAUDI ARABIA
E-mail: khmo.alshammari@uoh.edu.sa

Received: 16 February 2025
Accepted: 2 April 2025

*Corresponding author