

MODIFIED NUMERICAL METHOD FOR SOLVING FRACTIONAL ORDER INFECTIOUS DISEASE MODEL

(Kaedah Numerik Terubah untuk Penyelesaian Model Penyakit Berjangkit Berasaskan Terbitan Pecahan)

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ABSTRACT

In this paper, we consider numerical approximate solutions to space-time fractional differential equations involving Atangana–Baleanu–Caputo (ABC) sense fractional derivative operator. We then enhance the Toufik–Atangana numerical scheme through a complex combination of the trapezoidal scheme and the ABC fractional Euler scheme to improve the accuracy and robustness of our computational method. Finally, the proposed method is validated through numerical simulation of an infectious disease model. The model consists of a system of fractional differential equations with eight compartments, collectively called SIDARTHE: Susceptible (S), Infected (I), Diagnosed (D), Ailing (A), Recognized (R), Threatened (T), Healed (H), and Extinct (E). The numerical results presented graphically show that different fractional orders lead to different asymptotic behaviors.

Keywords: Atangana–Baleanu derivative; infectious disease model; fractional differential equation; numerical simulation; modified numerical scheme

ABSTRAK

Dalam kajian ini, kami membincangkan penyelesaian anggaran berangka terhadap persamaan pembezaan pecahan ruang-masa yang melibatkan pengoperasi pembeza dalam bentuk deria Atangana–Baleanu–Caputo (ABC). Kami seterusnya menambah baik skema berangka Toufik–Atangana dengan menggabungkan skema trapezoid dan skema Euler untuk terbitan pecahan bentuk ABC bagi meningkatkan ketepatan dan kualiti kaedah pengiraan kami. Akhirnya, cadangan kaedah ini disahkan melalui simulasi berangka terhadap satu model penyakit berjangkit. Model tersebut terdiri daripada sistem persamaan pembezaan pecahan dengan lapan kompartmen, secara kolektif dikenali sebagai SIDARTHE: *Susceptible* (S), *Infected* (I), *Diagnosis* (D), *Ailing* (A), *Recognized* (R), *Threatened* (T), *Healed* (H), and *Extinct* (E). Keputusan berangka yang dibentangkan secara grafik menunjukkan bahawa terbitan pecahan yang berlainan menghasilkan ragam bentuk asimptotik yang berbeza.

Kata kunci: terbitan Atangana–Baleanu; model penyakit berjangkit; persamaan pembezaan pecahan; simulasi berangka; skema berangka terubah

1. Introduction

Compared with traditional integer-order methods, fractional-order calculus, handles differentiation and integration of any order, providing a superior method for modelling real-world problems. Existing studies on mathematical models of infectious diseases often utilize ordinary derivatives (Ghosh *et al.* 2022a, b), Caputo (Caputo 1967; AlSadi *et al.* 2024; Li *et al.* 2017; Amouch & Karim 2023), Caputo-Fabrizio (Caputo & Fabrizio 2015; Baleanu *et al.* 2020), and Atangana–Baleanu (Atangana & Baleanu 2016; Butt *et al.* 2022) fractional derivatives to capture epidemic dynamics. For instance, Giordano *et al.* (2020) developed the SIDARTHE model to analyze the COVID-19 epidemic and compared its predictions with actual data in Italy through ordinary differential equations (ODEs). This model separates detected and undetected

infection cases and considers the different stages of disease progression. Further, Higazy (2020) introduced the SIDARTHE model with Caputo fractional derivatives to capture the memory effect in the progression of COVID-19 and explored the optimal control strategy. In addition, the fractional-order modelling approach using the Caputo-Fabrizio derivative was expanded by Zhao *et al.* (2024).

However, there exists a significant research gap in applying the Atangana–Baleanu–Caputo (ABC) derivative to study the fractional order of the SIDARTHE infectious disease model. To the best of our knowledge, the ABC operator is being applied to this eight-dimensional complex compartmental model for the first time. The Atangana–Baleanu fractional derivative in the Caputo sense (ABC) offers more advantages over the traditional Caputo derivative, primarily because of its non-local and non-singular kernel. Unlike the Caputo derivative, which relies on a power-law kernel with singularities (points where calculations can break down or become undefined), the ABC derivative uses the generalized Mittag-Leffler function, avoiding these issues and providing smoother, more accurate modelling of memory effects. Traditional Caputo derivatives struggle with certain natural phenomena due to their singularity, which can make models fail at specific points. The ABC derivative, developed by Atangana and Baleanu (2016), eliminates this by combining power and exponential decay laws, capturing memory effects more efficiently without singularities. Its non-local nature means that each state depends not only on the present but also on all past states, making it especially useful for complex systems with history-dependent behaviour (Jena *et al.* 2021). In this study, a modified numerical scheme is used to find numerical solutions. The modified method integrates trapezoidal and fractional Euler schemes with the Toufik–Atangana scheme (Toufik & Atangana 2017).

The structure of the paper is as follows: In Section 2, the SIDARTHE model is presented using ABC fractional-order derivatives. The existence and uniqueness of solutions are explored in Section 2.1. A detailed discussion of the modified Toufik–Atangana numerical scheme is provided in Section 3. Simulations and discussions on the SIDARTHE model are included in Section 4. Furthermore, a comprehensive error analysis is presented in Section 3.1. Finally, the paper is concluded, and the findings are summarized in Section 5.

2. Model Formulation

The mathematical formulation of the SIDARTHE infectious disease model is expressed in the sense of the ABC fractional-order compartment framework, as detailed in Giordano *et al.* (2020):

$$\begin{cases} {}^{ABC}D_t^\beta S(t) = -S(t)(aI(t) + bD(t) + cA(t) + dR(t)), \\ {}^{ABC}D_t^\beta I(t) = S(t)(aI(t) + bD(t) + cA(t) + dR(t)) - (e + z + g)I(t), \\ {}^{ABC}D_t^\beta D(t) = eI(t) - (h + r)D(t), \\ {}^{ABC}D_t^\beta A(t) = zI(t) - (\theta + m + k)A(t), \\ {}^{ABC}D_t^\beta R(t) = hD(t) + \theta A(t) - (v + x)R(t), \\ {}^{ABC}D_t^\beta T(t) = mA(t) + vR(t) - (\sigma + \tau)T(t), \\ {}^{ABC}D_t^\beta H(t) = gI(t) + rD(t) + kA(t) + xR(t) + \sigma T(t), \\ {}^{ABC}D_t^\beta E(t) = \tau T(t). \end{cases} \quad (1)$$

Taking the total population (N) as 100 million, the initial values of different population phases after normalization are:

$$\begin{aligned} (S(0), I(0), D(0), A(0), R(0), T(0), H(0), E(0)) \\ = \left(\left(1 - \frac{200}{N} - \frac{20}{N} - \frac{1}{N} - \frac{2}{N}\right), \frac{200}{N}, \frac{20}{N}, \frac{1}{N}, \frac{2}{N}, 0, 0, 0 \right). \end{aligned}$$

For convenience, let $\mathcal{Y}(t) = (S(t), I(t), \mathcal{D}(t), \mathcal{A}(t), \mathcal{R}(t), \mathcal{T}(t), \mathcal{H}(t), \mathcal{E}(t))$ and $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6, \Phi_7, \Phi_8)$, where $\Phi_1, \Phi_2, \dots, \Phi_8$ correspond to the right-hand terms of system (1). Specifically,

$$\Phi_1 = -S(t) (aI(t) + b\mathcal{D}(t) + c\mathcal{A}(t) + d\mathcal{R}(t)),$$

with similar definitions for Φ_2 through Φ_8 . Additionally, the function $\mathcal{Y}(t)$ belongs to $H^1(a, b)$, where H is a Hilbert space, $b > a$, and β lies in the interval $(0, 1]$. The Atangana–Baleanu–Caputo (ABC) fractional derivative as described in Atangana and Baleanu (2016) functions as follows:

$${}^{ABC}_0 D_t^\beta \mathcal{Y}(t) = \frac{Q(\beta)}{1-\beta} \int_0^t E_\beta \left(\frac{\beta}{\beta-1} (t-\lambda)^\beta \right) \mathcal{Y}'(\lambda) d\lambda, \quad (2)$$

where $Q(\beta)$ is a normalization function that depends on β , $Q(0) = Q(1) = 1$, and the formula for calculating $Q(\beta)$ is given by $Q(\beta) = 1 - \beta + \frac{\beta}{\Gamma(\beta)}$. $\Gamma(\cdot)$ denotes the Gamma function, and the Mittag-Leffler function is given by $E_\beta(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(1+i\beta)}$. The parameters, variables, and assumptions are listed in Table 1.

Table 1: Lists of variables and parameters

Symbol	Explanation (Higazy 2020)	Range
S	Susceptible: Not infected individuals	$S \geq 0$
I	Infected: Asymptomatic, undetermined individuals	$I \geq 0$
$\mathcal{D}$	Diagnosed: Asymptomatic, confirmed individuals	$\mathcal{D} \geq 0$
$\mathcal{A}$	Ailing: Symptomatic, undetermined individuals	$\mathcal{A} \geq 0$
$\mathcal{R}$	Recognized: Symptomatic, confirmed individuals	$\mathcal{R} \geq 0$
$\mathcal{T}$	Threatened: Severe symptoms cases	$\mathcal{T} \geq 0$
$\mathcal{H}$	Healed: Recovered individuals	$\mathcal{H} \geq 0$
$\mathcal{E}$	Extinct: Deceased individuals	$\mathcal{E} \geq 0$
a	Rate of infection after S and I interact	$0 \leq a < 1$
b	Rate of infection after S and $\mathcal{D}$ interact	$0 \leq b < 1$
c	Rate of infection after S and $\mathcal{A}$ interact	$0 \leq c < 1$
d	Rate of infection after S and $\mathcal{R}$ interact	$0 \leq d < 1$
e	Detection probability rate of asymptomatic carriers	$0 \leq e < 1$
z	Probability rate of symptoms in the undetected	$0 \leq z < 1$
θ	Detection probability rate of symptomatic carriers	$0 \leq \theta < 1$
h	Probability rate of symptoms being detected	$0 \leq h < 1$
m	Probability rate at which $\mathcal{A}$ have life-threatening symptoms	$0 \leq m < 1$
ν	Probability rate at which $\mathcal{R}$ have life-threatening symptoms	$0 \leq \nu < 1$
τ	Death rate for $\mathcal{T}$	$0 \leq \tau < 1$
g	Healing rate for I	$0 \leq g < 1$
k	Healing rate for $\mathcal{A}$	$0 \leq k < 1$
x	Healing rate for $\mathcal{R}$	$0 \leq x < 1$
r	Healing rate for $\mathcal{D}$	$0 \leq r < 1$
σ	Healing rate for $\mathcal{T}$	$0 \leq \sigma < 1$

To show that Eq. (1) can be reduced to an integral form as in Eq. (3) by using Eq. (2), we proceed by applying the ABC fractional derivative definition in Eq. (2) to each term in Eq. (1), we convert each differential equation into an integral form, expressing each compartmental

equation in terms of fractional integration as:

$$\frac{Q(\beta)}{1-\beta} \int_0^t E_\beta \left(\frac{\beta}{\beta-1} (t-\lambda)^\beta \right) \mathcal{Y}'(\lambda) d\lambda = \Phi(t, \mathcal{Y}(t)),$$

where $\Phi(t, \mathcal{Y}(t))$ represents the right-hand side of each equation in Eq. (1).

For each compartment $\mathcal{S}(t), \mathcal{I}(t), \mathcal{D}(t), \mathcal{A}(t), \mathcal{R}(t), \mathcal{T}(t), \mathcal{H}(t)$ and $\mathcal{E}(t)$, using fundamental theorem of fractional calculus, as described in Toufik and Atangana (2017), yields the following equation:

$$\mathcal{Y}(t) - \mathcal{Y}(0) = \frac{1-\beta}{Q(\beta)} \Phi(t, \mathcal{Y}(t)) + \frac{\beta}{Q(\beta)\Gamma(\beta)} \int_0^t \Phi(\lambda, \mathcal{Y}(\lambda)) (t-\lambda)^{\beta-1} d\lambda. \quad (3)$$

This integral equation, applied with initial conditions $\mathcal{Y}(0) = (\mathcal{S}(0), \mathcal{I}(0), \mathcal{D}(0), \mathcal{A}(0), \mathcal{R}(0), \mathcal{T}(0), \mathcal{H}(0), \mathcal{E}(0))$, describes the fractional dynamics of the SIDARTHE model.

2.1. Existence and uniqueness of solutions

In this section, we apply fixed-point methods to establish the existence and uniqueness of solutions in the formulated SIDARTHE model (1).

Assumption 2.1. The functions $\mathcal{S}(t), \check{\mathcal{S}}(t), \mathcal{I}(t), \check{\mathcal{I}}(t), \mathcal{D}(t), \check{\mathcal{D}}(t), \mathcal{A}(t), \check{\mathcal{A}}(t), \mathcal{R}(t), \check{\mathcal{R}}(t), \mathcal{T}(t), \check{\mathcal{T}}(t), \mathcal{H}(t), \check{\mathcal{H}}(t), \mathcal{E}(t), \check{\mathcal{E}}(t)$ belong to a Banach space $\mathcal{U} = [0, p]$ and are continuous, where $p \in \mathbb{R}$. Their supremum norms $\|\mathcal{S}\|, \|\mathcal{I}\|, \|\mathcal{D}\|, \|\mathcal{A}\|, \|\mathcal{R}\|, \|\mathcal{T}\|, \|\mathcal{H}\|$ and $\|\mathcal{E}\|$ are bounded above by non-negative constants κ_i , for $i \in \mathcal{N}_1^8$, where $\mathcal{N}_1^8 = \{1, 2, 3, \dots, 8\}$, is the set of indices from 1 to 8.

Theorem 2.2. Suppose that Assumption 2.1 is satisfied and that $\mu_i < 1$ for $i \in \mathcal{N}_1^8$. Then, for each $i \in \mathcal{N}_1^8$, the Φ_i function satisfies the Lipschitz condition for the ABC derivative operator.

Proof. First, we prove that $\Phi_1(t, \mathcal{S}(t))$ satisfies the Lipschitz condition. For $\mathcal{S}(t)$ and $\check{\mathcal{S}}(t)$, we get

$$\begin{aligned} \|\Phi_1(t, \mathcal{S}) - \Phi_1(t, \check{\mathcal{S}})\| &= \|(-a\mathcal{I}(t) - b\mathcal{D}(t) - c\mathcal{A}(t) - d\mathcal{R}(t))(\mathcal{S}(t) - \check{\mathcal{S}}(t))\| \\ &\leq \| -a\mathcal{I}(t) - b\mathcal{D}(t) - c\mathcal{A}(t) - d\mathcal{R}(t) \| \|\mathcal{S}(t) - \check{\mathcal{S}}(t)\| \\ &\leq (a\|\mathcal{I}(t)\| + b\|\mathcal{D}(t)\| + c\|\mathcal{A}(t)\| + d\|\mathcal{R}(t)\|) \|\mathcal{S}(t) - \check{\mathcal{S}}(t)\| \\ &\leq \mu_1 \|\mathcal{S}(t) - \check{\mathcal{S}}(t)\|, \end{aligned} \quad (4)$$

where $\mu_1 = a\kappa_2 + b\kappa_3 + c\kappa_4 + d\kappa_5$. It has been demonstrated that Φ_1 satisfies the Lipschitz condition, where μ_1 is the Lipschitz constant and Φ_1 serves as the Lipschitz function for $\mathcal{S}$. The Lipschitz conditions for the other seven functions can also be verified in a similar manner. Therefore, the following expression is presented:

$$\|\Phi_i(t, \mathcal{Y}_i(t)) - \Phi_i(t, \check{\mathcal{Y}}_i(t))\| \leq \mu_i \|\mathcal{Y}_i(t) - \check{\mathcal{Y}}_i(t)\|,$$

where $\mu_2 = m\kappa_1 + e + z + g, \mu_3 = h + r, \mu_4 = \theta + m + k, \mu_5 = v + x, \mu_6 = \delta + \tau, \mu_7 = 0, \mu_8 = 0$. Hence, for each $i \in \mathcal{N}_1^8, \mathcal{Y}(t) = (\mathcal{S}(t), \mathcal{I}(t), \mathcal{D}(t), \mathcal{A}(t), \mathcal{R}(t), \mathcal{T}(t), \mathcal{H}(t), \mathcal{E}(t))$ and $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6, \Phi_7, \Phi_8)$, the Lipschitz condition for the function Φ_i is observed, and the corresponding constants $\mu_i < 1$. The proof is complete. $\square$

When processed recursively, Eq. (3) can be reformulated as follows:

$$\mathcal{Y}^{(n+1)}(t) - \mathcal{Y}(0) = \frac{1-\beta}{Q(\beta)} \Phi(t, \mathcal{Y}^{(n)}(t)) + \frac{\beta}{Q(\beta)\Gamma(\beta)} \int_0^t \Phi(\lambda, \mathcal{Y}^{(n)}(\lambda))(t-\lambda)^{\beta-1} d\lambda,$$

where the superscripts in $\mathcal{Y}^{(n)}$ and $\mathcal{Y}^{(n+1)}$ indicate the iteration numbers, and as n goes to infinity, $\mathcal{Y}^{(n)}(t)$ should converge to $\mathcal{Y}(t)$, the actual solution of the integral equation.

Theorem 2.3. *Model (1) admits a solution if the following conditions are satisfied for $i \in \mathcal{N}_1^8$ and*

$$\psi = \max \mu_i < 1. \quad (5)$$

Proof. Consider the following functions:

$$\mathcal{E}_{\mathcal{Y}_i, n}(t) = \mathcal{Y}_i^{(n+1)}(t) - \mathcal{Y}_i(t), \quad (6)$$

where n is the step of the iteration and $\mathcal{E}_{\mathcal{Y}_i, n}(t)$ denotes the difference between the $(n+1)$ -th order approximation and the actual value of the respective i -th compartment at time t . Hence, we have

$$\begin{aligned} \mathcal{E}_{\mathcal{Y}_i, n}(t) &= \frac{1-\beta}{Q(\beta)} \left(\Phi_i(t, \mathcal{Y}_i^{(n)}(t)) - \Phi(t, \mathcal{Y}_i(t)) \right) \\ &\quad + \frac{\beta}{Q(\beta)\Gamma(\beta)} \int_0^t \left(\Phi(\lambda, \mathcal{Y}_i^{(n)}(\lambda)) - \Phi(\lambda, \mathcal{Y}_i(\lambda)) \right) (t-\lambda)^{\beta-1} d\lambda, \end{aligned}$$

and subsequently, it follows that

$$\begin{aligned} \|\mathcal{E}_{\mathcal{Y}_i, n}(t)\| &= \frac{1-\beta}{Q(\beta)} \left\| \Phi_i(t, \mathcal{Y}_i^{(n)}(t)) - \Phi(t, \mathcal{Y}_i(t)) \right\| \\ &\quad + \frac{\beta}{Q(\beta)\Gamma(\beta)} \int_0^t \left\| \Phi(\lambda, \mathcal{Y}_i^{(n)}(\lambda)) - \Phi(\lambda, \mathcal{Y}_i(\lambda)) \right\| (t-\lambda)^{\beta-1} d\lambda. \end{aligned}$$

Therefore, applying the triangle inequality to the previous equations and incorporating the recursive hypothesis, we obtain

$$\|\mathcal{E}_{\mathcal{Y}_i, n}(t)\| \leq \left(\frac{1-\beta}{Q(\beta)} \mu_i + \frac{t^\beta}{Q(\beta)\Gamma(\beta)} \mu_i \right) \|\mathcal{Y}_i^{(n)}(t) - \mathcal{Y}_i(t)\|.$$

Thus,

$$\|\mathcal{E}_{\mathcal{Y}_i, n}(t)\| \leq \left(\frac{1-\beta}{Q(\beta)} + \frac{t_0^\beta}{Q(\beta)\Gamma(\beta)} \right)^n \mu_i^n \|\mathcal{Y}_i^{(1)}(t) - \mathcal{Y}_i(t)\|, \quad (7)$$

where i belongs to the set $\mathcal{N}_1^8$, which includes the compartments $\mathcal{S}(t)$, $\mathcal{I}(t)$, $\mathcal{D}(t)$, $\mathcal{A}(t)$, $\mathcal{R}(t)$, $\mathcal{T}(t)$, $\mathcal{H}(t)$, and $\mathcal{E}(t)$. Therefore, the functions discussed above show that as n approaches infinity, $\mathcal{E}_{\mathcal{Y}_i, n}$ tends to zero for each i in $\mathcal{N}_1^8$, provided that $\mu_i < 1$. This concludes the proof. $\square$

Theorem 2.4. *Under Assumption 2.1 system (1) admits only, a unique solution, provided that*

for $i \in \mathcal{N}_1^8$,

$$\left(\frac{1-\beta}{Q(\beta)} + \frac{t^\beta}{Q(\beta)\Gamma(\beta)} \right) \mu_i \leq 1. \quad (8)$$

Proof. We assume that there is an alternative solution $\check{S}(t), \check{I}(t), \check{D}(t), \check{A}(t), \check{R}(t), \check{T}(t), \check{H}(t), \check{E}(t)$ with the same initial values, represented as $\check{S}(0), \check{I}(0), \check{D}(0), \check{A}(0), \check{R}(0), \check{T}(0), \check{H}(0), \check{E}(0)$. In particular,

$$\check{S}(t) - \check{S}(0) = \frac{1-\beta}{Q(\beta)} \Phi_1(t, \check{S}(t)) + \frac{\beta}{Q(\beta)\Gamma(\beta)} \int_0^t \Phi_1(\lambda, \check{S}(\lambda))(t-\lambda)^{\beta-1} d\lambda.$$

Now, we write

$$\begin{aligned} \|S(t) - \check{S}(t)\| &= \frac{1-\beta}{Q(\beta)} \|\Phi_1(t, S(t)) - \Phi_1(t, \check{S}(t))\| \\ &\quad + \frac{\beta}{Q(\beta)\Gamma(\beta)} \int_0^t \|(\Phi_1(\lambda, S(\lambda)) - \Phi_1(\lambda, \check{S}(\lambda))\|(t-\lambda)^{\beta-1} d\lambda. \end{aligned}$$

Therefore, by virtue of using Eq. (4), we obtain

$$\|S(t) - \check{S}(t)\| \leq \left(\frac{1-\beta}{Q(\beta)} + \frac{t^\beta}{Q(\beta)\Gamma(\beta)} \right) \mu_1 \|S(t) - \check{S}(t)\|.$$

Finally,

$$\left[1 - \left(\frac{1-\beta}{Q(\beta)} + \frac{t^\beta}{Q(\beta)\Gamma(\beta)} \right) \mu_1 \right] \|S(t) - \check{S}(t)\| \leq 0. \quad (9)$$

Clearly, (9) is valid when the norm $\|S(t) - \check{S}(t)\| = 0$. This indicates that $S(t)$ and $\check{S}(t)$ are identical. This confirms the uniqueness of the solution for $S(t)$. Similarly, the uniqueness of the solutions for the functions $I(t), D(t), A(t), R(t), T(t), H(t)$, and $E(t)$ can be established. Therefore, the system described in (1) possesses a unique solution. $\square$

3. Modified Toufik–Atangana Numerical Scheme

The main contribution of this study lies in the advancement of the numerical techniques originally introduced by Toufik and Atangana (2017). This enhancement includes the integration of the classical trapezoidal scheme with the fractional Euler method. Consider the initial value problem of the ABC operator:

$$\begin{cases} {}^{ABC}D_t^\beta \eta(t) = f(t, \eta(t)), \\ \eta(0) = \eta_0. \end{cases} \quad (10)$$

By applying the fundamental theorem of fractional calculus, Eq. (10) can be transformed into a fractional integral equation:

$$\eta(t) - \eta(0) = \frac{1-\beta}{Q(\beta)} f(t, \eta(t)) + \frac{\beta}{Q(\beta)\Gamma(\beta)} \int_0^t f(\tau, \eta(\tau))(t-\tau)^{\beta-1} d\tau. \quad (11)$$

For a given instance t_{n+1} where n takes values 1, 2, 3, and so on, and $t_n = t_0 + nh$ with $t_0 = 0$, Eq. (11) can be reformulated as follows:

$$\begin{aligned}\eta(t_{n+1}) - \eta(0) &= \frac{(1-\beta)}{Q(\beta)} f(t_n, \eta(t_n)) + \frac{\beta}{Q(\beta)\Gamma(\beta)} \int_0^{t_{n+1}} f(\tau, \eta(\tau))(t_{n+1} - \tau)^{\beta-1} d\tau \\ &= \frac{(1-\beta)}{Q(\beta)} f(t_n, \eta(t_n)) + \frac{\beta}{Q(\beta)\Gamma(\beta)} \int_{t_0}^{t_1} f(\tau, \eta(\tau))(t_{n+1} - \tau)^{\beta-1} d\tau \\ &\quad + \frac{\beta}{Q(\beta)\Gamma(\beta)} \sum_{k=1}^n \int_{t_k}^{t_{k+1}} f(\tau, \eta(\tau))(t_{n+1} - \tau)^{\beta-1} d\tau.\end{aligned}\quad (12)$$

Employing the trapezoidal rule over the interval $[t_0, t_1]$, the following integral can be approximated as:

$$\int_{t_0}^{t_1} f(\tau, \eta(\tau))(t_{n+1} - \tau)^{\beta-1} d\tau = \frac{h^\beta}{2} [f(t_0, \eta_0)(n+1)^{\beta-1} + f(t_1, \eta_1)n^{\beta-1}]. \quad (13)$$

Furthermore, in order to estimate the value of the second integral term, the function $f(\tau, \eta(\tau))$ is approximated by applying a two-step Lagrange polynomial interpolation over the interval $[t_{k-1}, t_k]$. Specifically, the function $f(\tau, \eta(\tau))$ can be approximated as:

$$f(\tau, \eta(\tau)) \approx \frac{f(t_k, \eta_k)}{h}(\tau - t_{k-1}) - \frac{f(t_{k-1}, \eta_{k-1})}{h}(\tau - t_k). \quad (14)$$

Let $\eta(t_{n+1})$ be denoted as η_{n+1} . Then,

$$\begin{aligned}\eta_{n+1} - \eta_0 &= \frac{1-\beta}{Q(\beta)} f(t_n, \eta_n) + \frac{\beta}{Q(\beta)\Gamma(\beta)} \frac{h^\beta}{2} [f(t_0, \eta_0)(n+1)^{\beta-1} + f(t_1, \eta_1)n^{\beta-1}] \\ &\quad + \frac{\beta}{Q(\beta)\Gamma(\beta)h} \sum_{k=1}^n f(t_k, \eta_k) \int_{t_k}^{t_{k+1}} (\tau - t_{k-1})(t_{n+1} - \tau)^{\beta-1} d\tau \\ &\quad - \frac{\beta}{Q(\beta)\Gamma(\beta)h} \sum_{k=1}^n f(t_{k-1}, \eta_{k-1}) \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+1} - \tau)^{\beta-1} d\tau.\end{aligned}\quad (15)$$

For convenience, we denote:

$$\begin{aligned}\gamma_{\beta,k,1} &= \int_{t_k}^{t_{k+1}} (\tau - t_{k-1})(t_{n+1} - \tau)^{\beta-1} d\tau \\ &= h^{\beta+1} \frac{(n+1-k)^\beta(n-k+2+\beta) - (n-k)^\beta(n-k+2+2\beta)}{\beta(\beta+1)}.\end{aligned}\quad (16)$$

Similarly,

$$\gamma_{\beta,k,2} = \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+1} - \tau)^{\beta-1} d\tau = h^{\beta+1} \frac{(n+1-k)^{\beta+1} - (n-k)^\beta(n-k+1+\beta)}{\beta(\beta+1)}, \quad (17)$$

where h is the discretization step size and β is the fractional-order and $t_{n+1} - t_k = (n - k + 1)h$, $t_{n+1} - t_{k+1} = (n - k)h$.

After integrating Eqs.(16) and (17) and then using Eq. (15), we obtain the following expression:

$$\begin{aligned} \eta_{n+1} = & \eta_0 + \frac{(1-\beta)}{Q(\beta)} f(t_n, \eta(t_n)) + \frac{\beta}{Q(\beta)\Gamma(\beta)} \frac{h^\beta}{2} [f(t_0, \eta_0)(n+1)^{\beta-1} + f(t_1, \eta_1)n^{\beta-1}] \\ & + \frac{\beta}{Q(\beta)\Gamma(\beta+2)} h^\beta \sum_{k=1}^n \left[f(t_k, \eta_k) \left((n+1-k)^\beta (n-k+2+\beta) - (n-k)^\beta (n-k+2+2\beta) \right) \right] \\ & - \frac{\beta}{Q(\beta)\Gamma(\beta+2)} h^\beta \sum_{k=1}^n \left[f(t_{k-1}, \eta_{k-1}) \left((n+1-k)^{\beta+1} - (n-k)^\beta (n-k+1+\beta) \right) \right]. \end{aligned} \quad (18)$$

To determine η_1 , we will use the fractional Euler method in the ABC sense during the initial iteration as follows:

$$\eta_1 = \eta_0 + \frac{(1-\beta)}{Q(\beta)} f(t_0, \eta_0) + \frac{\beta h^\beta}{Q(\beta)\Gamma(\beta+1)} f(t_0, \eta_0).$$

We adapt our approach in two stages. First, we apply the trapezoidal method to the initial interval $[t_0, t_1]$. Then, we employ the fractional Euler technique for the ABC framework. This dual-strategy modification ensures a more precise and reliable numerical approximation.

3.1. Comprehensive analysis of error

To assess the precision and efficacy of the proposed method, we scrutinize the fractional differential equation as outlined in Toufik and Atangana (2017), using the ABC operator:

$$\begin{cases} {}^{ABC}_0 D_t^\beta \eta(t) = t^2, \\ \eta(0) = 0. \end{cases} \quad (19)$$

We have identified numerical discrepancies with the exact solution presented by Toufik and Atangana (2017). Specifically, the exact solution of Eq. (19) is

$$\eta(t) = \frac{1-\beta}{Q(\beta)} t^2 + \frac{2t^{\beta+2}}{Q(\beta)\Gamma(\beta)} \left[\frac{1}{\beta^2 + 3\beta + 2} \right]. \quad (20)$$

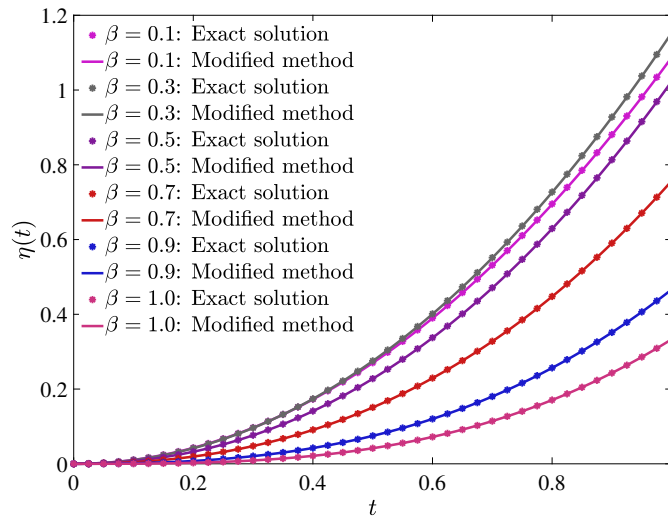
Utilizing the proposed numerical scheme presented in this paper, the approximation error is obtained as follows:

$$\|R_\beta\| = \|\eta_{\text{approximate}}(t) - \eta_{\text{exact}}(t)\|_\infty.$$

We compared the error of the Toufik–Atangana numerical scheme with the error of the proposed numerical scheme, which is based on Eq. (20), as shown in Table 2. Analyzing data in Table 2, it can be seen that the modified scheme can improve the error as the step size h and the value of fractional-order β increases. In addition, the comparison of the modified method with the exact solutions shows good agreement, as shown in Figure 1.

Table 2: Error comparison between the numerical scheme proposed by Toufik and Atangana (2017) and the proposed numerical scheme (the present work)

h	β	Toufik–Atangana scheme	Modified scheme
0.01	0.30	0.069947	0.069946
0.10	0.30	0.697621	0.697307
0.01	0.75	0.023395	0.023394
0.10	0.75	0.251555	0.250923
0.01	0.80	0.018258	0.018257
0.05	0.90	0.049363	0.049288
0.10	0.95	0.071348	0.070816
0.10	0.99	0.040342	0.039835

Figure 1: Comparison of the modified numerical scheme with exact solutions for different values of β

4. Numerical Simulation

For the numerical simulations, the parameter values for the system under consideration, as taken from Giordano *et al.* (2020), are as follows: $a = 0.57$, $b = 0.0114$, $c = 0.456$, $d = 0.0114$, $e = 0.171$, $z = 0.1254$, $\theta = 0.3705$, $h = 0.1254$, $m = 0.0171$, $\nu = 0.0274$, $\tau = 0.01$, $g = 0.0342$, $k = 0.0171$, $x = 0.0171$, $r = 0.0342$, and $\sigma = 0.0171$. There is no exact solution for the non-linear fractional SIDARTHE model. Therefore, the validity of our modified scheme is assessed by comparison with the existing ode45 (Zhao *et al.* 2024) numerical results. The ode45 method is selected as the benchmark for comparison. The numerical results obtained using our method and ode45 are compared in Table 3. For example, at all values of t , the extinct population ($\mathcal{E}(t)$) is lower than the values obtained using ode45. Similarly, for other variables, most values are either lower or closely aligned with the ode45 results. This confirms the excellent performance of our modified method. Figure 2 displays the time history of each population stage for the ABC operator with different fractional derivative orders. Based on the results presented in all the figures in this section, it can be stated that a decrease in the fractional derivative order β leads to a reduction in the number of infected cases and delays the time required to reach the maximum number in each population stage. It also makes the curves of all population stages flatter. These demonstrate the notable memory property inherent to fractional-order dynamical systems.

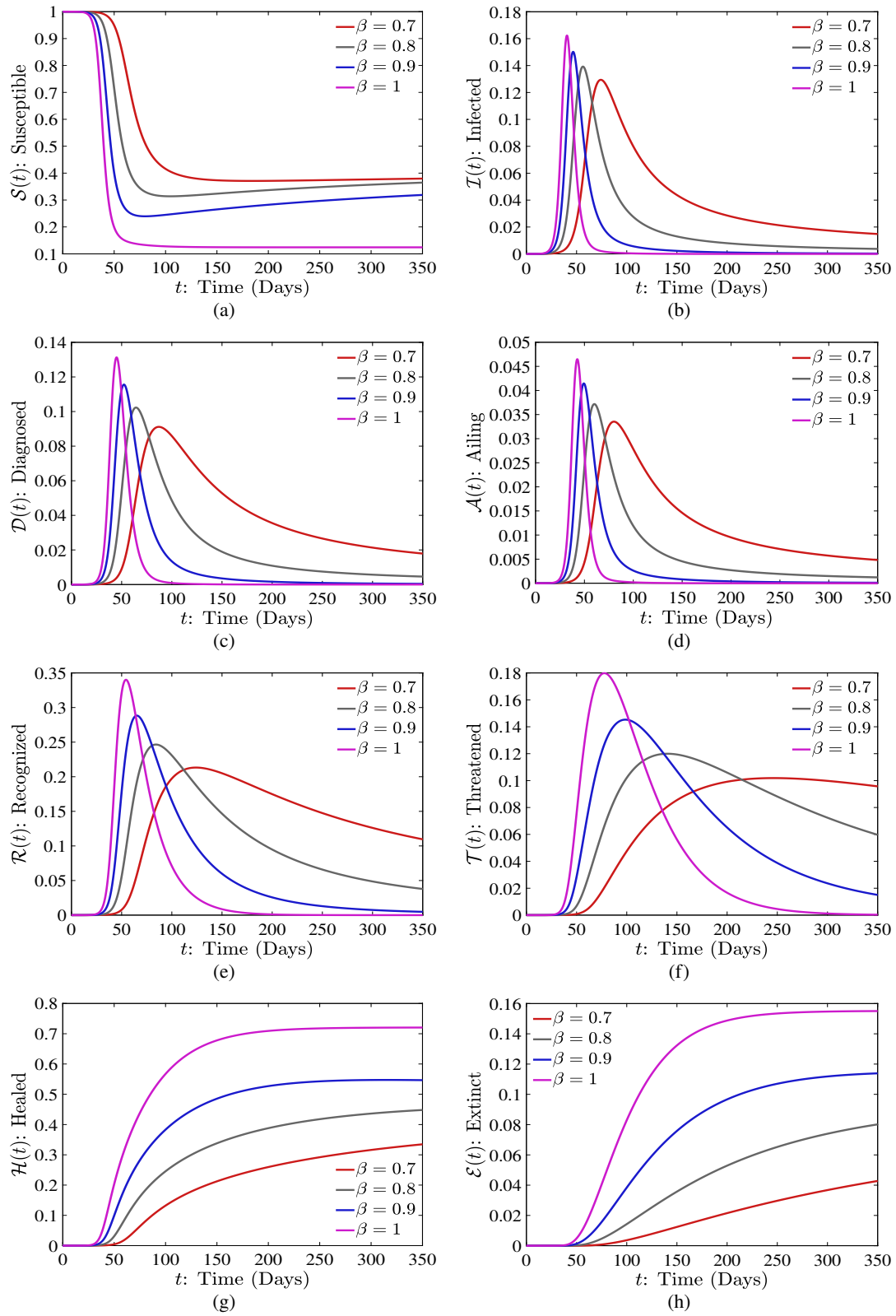


Figure 2: Temporal dynamics of states for the cases of $\beta = 0.7, 0.8, 0.9, 1$ for: (a) susceptible (S), (b) infected (I), (c) diagnosed ($\mathcal{D}$), (d) ailing ($\mathcal{A}$), (e) recognized ($\mathcal{R}$), (f) threatened ($\mathcal{T}$), (g) healed ($\mathcal{H}$), and (h) extinct ($\mathcal{E}$).

Table 3: Comparison of the numerical results obtained using ode45, as presented by Zhao *et al.* (2024), and the numerical results obtained using the present modified numerical scheme for various compartments at $\beta = 1$ over time (t) in days.

Compartment	Methods	$t = 0$	$t = 20$	$t = 40$	$t = 60$	$t = 80$
$\mathcal{S}(t)$	ode45	1	0.9975	0.4845	0.1501	0.1329
	Present study	1	0.9976	0.4905	0.1503	0.1329
$\mathcal{I}(t)$	ode45	2×10^{-6}	0.0012	0.1620	0.0113	0.0015
	Present study	2×10^{-6}	0.0012	0.1618	0.0115	0.0015
$\mathcal{D}(t)$	ode45	2×10^{-7}	4.31×10^{-4}	0.1021	0.0419	0.0043
	Present study	2×10^{-7}	4.17×10^{-4}	0.1009	0.0424	0.0043
$\mathcal{A}(t)$	ode45	1×10^{-8}	2.09×10^{-4}	0.0422	0.0058	5.45×10^{-4}
	Present study	1×10^{-8}	2.03×10^{-4}	0.0418	0.0059	5.50×10^{-4}
$\mathcal{R}(t)$	ode45	2×10^{-8}	3.56×10^{-4}	0.1246	0.3157	0.1611
	Present study	2×10^{-8}	3.45×10^{-4}	0.1222	0.3182	0.1618
$\mathcal{T}(t)$	ode45	0	3.77×10^{-5}	0.0163	0.1401	0.1793
	Present study	0	3.65×10^{-5}	0.0159	0.1395	0.1793
$\mathcal{H}(t)$	ode45	0	2.05×10^{-4}	0.0676	0.3173	0.4708
	Present study	0	1.98×10^{-4}	0.0663	0.3163	0.4702
$\mathcal{E}(t)$	ode45	0	1.72×10^{-6}	6.11×10^{-4}	0.0159	0.0497
	Present study	0	1.09×10^{-6}	5.94×10^{-4}	0.0158	0.0495

5. Conclusion

This paper studies the SIDARTHE model using Atangana–Baleanu–Caputo fractional derivative. We use a fixed-point approach with the ABC operator to establish the existence and uniqueness of solutions. The modified Toufik–Atangana scheme helped us find numerical solutions and understand the behavior of the system. Our research shows that the Atangana–Baleanu fractional derivative can capture the memory effect of disease spread better than traditional models. The spread of the disease can be significantly controlled by reducing the infection rate and increasing the detection of asymptomatic cases. The results obtained in this study show that the application of fractional calculus in epidemiology is highly relevant and can lead to better disease control strategies.

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