

SOME BOUNDS FOR THE MINIMUM EDGE DOMINATING ENERGY OF GRAPHS

(Beberapa Batas untuk Tenaga Dominasi Sisi Minimum dalam Graf)

FATEME MOVAHEDI, MOHAMAD HADI AKHBARI, ROSLAN HASNI*,
NABEEL EZULDDIN ARIF & SUMARNI ABU BAKAR

ABSTRACT

The minimum edge dominating energy, denoted by $EE_F(G)$, of a graph G is the sum of the absolute values of eigenvalues of the minimum edge dominating matrix with respect to a minimum edge dominating set F of the graph G . In this study, some bounds for the minimum edge dominating energy are obtained.

Keywords: eigenvalue; minimum edge dominating energy; line graph

ABSTRAK

Tenaga dominasi sisi minimum, ditandakan sebagai $EE_F(G)$, bagi suatu graf ialah jumlah nilai nyata bagi nilai eigen daripada matrik dominasi sisi minimum terhadap satu set dominasi sisi minimum F bagi graf G . Dalam kertas ini, beberapa batas bagi tenaga dominasi sisi minimum diperoleh.

Kata kunci: tenaga dominasi sisi minimum; nilai eigen; graf garisan

1. Introduction

Consider $G = (V_G, E_G)$ as a simple graph with n vertices and m edges. The set $N_G(t) = \{s \in V_G | ts \in E_G\}$ is called the neighborhood of vertex t in graph G . The number of edges incident to vertex u in G is denoted $deg_G(u)$.

Assume that $V_G = \{v_1, \dots, v_n\}$ and $E_G = \{e_1, \dots, e_m\}$. We consider $v_i \sim v_j$ for $v_i v_j \in E_G$. The adjacency matrix of G , $A(G) = (\omega_{ij})$ is an $n \times n$ matrix, where $\omega_{ij} = 1$ if $v_i \sim v_j$ and $\omega_{ij} = 0$ otherwise. Suppose that the eigenvalues of the graph G are denoted by ξ_i for $1 \leq i \leq n$. The concept of graph energy was introduced in 1978 Gutman (1978) and is defined as $E(G) = \sum_{i=1}^n |\xi_i|$. A comprehensive discussion on the results related to graph energy can be found in Li *et al.* (2012).

The line graph L_G of a graph G is formed by representing each edge of G as a vertex in L_G . Two vertices in L_G are linked if and only if their corresponding edges in G have a common vertex Harary (1969). The concept of edge energy of G , denoted by $EE(G)$, was introduced in Bozkurt and Bozkurt (2014) as the graph energy of the line graph L_G .

A subset $\mathcal{D} \subseteq V_G$ is called the dominating set if every vertex in $V_G \setminus \mathcal{D}$ is adjacent to at least one vertex in $\mathcal{D}$. A dominating set with the smallest possible size is referred to as a minimum dominating set Haynes *et al.* (1998). Similarly, a set $\mathcal{F} \subseteq E_G$ is an edge dominating set if every edge in $E_G \setminus \mathcal{F}$ is adjacent to at least one edge in $\mathcal{F}$. The edge domination number of G is defined as the minimum size among all edge dominating sets of G Gupta (1969). It is important to note that an edge dominating set of G corresponds to a dominating set in its line graph L_G , and vice versa.

The minimum edge dominating matrix of G is defined by $A_{\mathcal{F}}(G) := (\omega_{ij})$ in Akhbari *et al.* (2020) where

$$\omega_{ij} = \begin{cases} 1 & \text{if } e_i \sim e_j, \\ 1 & \text{if } i = j \text{ and } e_i \in \mathcal{F}, \\ 0 & \text{otherwise.} \end{cases}$$

Based on the above matrix, in Akhbari *et al.* (2020), the minimum edge dominating energy of G is defined as

$$EE_{\mathcal{F}}(G) = \sum_{i=1}^m |\xi_i|,$$

where ξ_i for $1 \leq i \leq m$ are the eigenvalues of $A_{\mathcal{F}}(G)$. Clearly, $\sum_{i=1}^m \xi_i = |\mathcal{F}|$. In Akhbari *et al.* (2020), for eigenvalues of $A_{\mathcal{F}}(G)$, the following result is obtained.

$$\sum_{i=1}^m \xi_i^2 = |\mathcal{F}| - 2m + \sum_{i=1}^n \deg(v_i)^2. \quad (1)$$

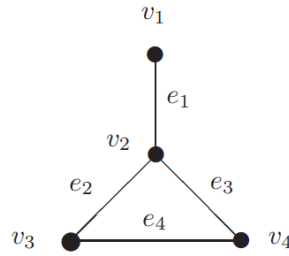


Figure 1: The graph G consists of vertices v_i and edges e_j for $1 \leq i, j \leq 4$.

Example 1.1. Consider the graph G depicted in Figure 1, which has $V_G = \{v_1, v_2, v_3, v_4\}$ and $E_G = \{e_1, e_2, e_3, e_4\}$. With considering $\mathcal{F} = \{e_2, e_3\}$, $A_{\mathcal{F}}(G)$ is as follows

$$A_{\mathcal{F}}(G) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

Hence, the characteristic polynomial of $A_G(G)$ is as

$$P(G, \xi) = \xi^4 - 2\xi^3 - 3\xi^2 + 2\xi.$$

Therefore, the eigenvalues of $A_{\mathcal{F}}(G)$ in this graph are

$$\xi_1 = 0, \quad \xi_2 = 0.529317, \quad \xi_3 = -1.3429, \quad \xi_4 = 2.81361.$$

Consequently,

$$EE_{\mathcal{F}}(G) = \sum_{i=1}^4 |\xi_i| = 4.68585.$$

Movahedi (2020) explored the relationships between $EE_{\mathcal{F}}(G)$ and its graph energies. Additional bounds on $E_{\mathcal{F}}$ of graphs were examined in Movahedi and Akhbari (2022). Furthermore, Movahedi (2021) established certain bounds for $EE_{\mathcal{F}}$ of subgraphs of a given graph.

In Movahedi (2024), several results regarding the eigenvalues and energy of $\rho_{\mathcal{F}}$ in caterpillar graphs were presented. For further findings on $EE_{\mathcal{F}}$ of graphs, see Chokani *et al.* (2023a, b, c). In this paper, as a comparative analysis of these previous results, we focus on establishing new bounds for $EE_{\mathcal{F}}$ of graphs.

In Movahedi (2024), some results of the eigenvalues and energy of $\rho_{\mathcal{F}}$ are obtained in caterpillars. For more results on $EE_{\mathcal{F}}$ of graphs, refer to Chokani *et al.* (2023a, b, c). In this paper, as a comparative study of the above results, we are interested in finding the other new bounds of $EE_{\mathcal{F}}$ of graphs.

Throughout this paper, we will use the following notations consistently. These notations will be used uniformly in all subsequent sections to maintain clarity and reduce redundancy. Consider G as a simple graph of order n and size m with a minimum edge dominating set $\mathcal{F}$ that $|\mathcal{F}| = k$. We consider ξ_i the eigenvalues of $A_{\mathcal{F}}(G)$ for $1 \leq i \leq m$. The degree of the vertex v_i in G is denoted by d_i .

2. Preliminaries

This section presents some previously established results that will be essential for the following section.

Lemma 2.1. (So 1994) Suppose $AB = BA$ where A and B are Hermitian matrices of order n . Then, there are a and b of $\{1, \dots, n\}$ which for all $k = 1, \dots, n$

$$\xi_k(A + B) = \xi_{a(k)}(A) + \xi_{b(k)}(B).$$

Lemma 2.2. (Das & Mojallal 2016) Let $\xi_i(G)$ be the eigenvalue of G for $1 \leq i \leq n$. Then

$$E(G) = 2 \sum_{i=1}^{\kappa^+} \xi_i(G) = 2 \max_{1 \leq k \leq n} \sum_{i=1}^k \xi_i(G),$$

where κ^+ represents the count of positive eigenvalues of $A(G)$.

Lemma 2.3. (So 1994) Suppose A and B are $n \times n$ Hermitian matrices and $C = A + B$. Then

$$\begin{aligned} \xi_i(C) &\leq \xi_j(A) + \xi_{i-j+1}(B) \quad 1 \leq j \leq i \leq n, \\ \xi_i(C) &\geq \xi_j(A) + \xi_{i-j+n}(B) \quad 1 \leq i \leq j \leq n, \end{aligned}$$

where $\xi_i(X)$ is the i th largest eigenvalue of the matrix X ($X = A, B, C$).

Lemma 2.4. (Movahedi 2020) For a graph G ,

$$EE_{\mathcal{F}}(G) \leq EE(G) + k.$$

Lemma 2.5. (Das *et al.* 2016) If T is a tree, then $E(L_T) \leq 2(|V(L_T)| - 1)$.

Lemma 2.6. (Cvetković *et al.* 1995) Assume G is a graph with r , q and p representing the number of eigenvalues that are equal to, less than, and greater than zero, respectively. Then

$$\alpha(G) \leq r + \min\{p, q\},$$

where $\alpha(G)$ is the independence number of G .

Lemma 2.7. (Mitrinović *et al.* 1993; Radon 1913) Let $\rho = (\rho_i)_{i=1}^n$ and $\mu = (\mu_i)_{i=1}^n$ be two sequences of positive numbers. Then

i) for any $r \geq 0$,

$$\sum_{i=1}^n \frac{\rho_i^{r+1}}{\mu_i^r} \geq \frac{\left(\sum_{i=1}^n \rho_i\right)^{r+1}}{\left(\sum_{i=1}^n \mu_i\right)^r},$$

ii) for any real number $r \leq 0$ or $r \geq 1$,

$$\sum_{i=1}^n \mu_i \rho_i^r \geq \sum_{i=1}^n \mu_i \left(\frac{\sum_{i=1}^n \mu_i \rho_i}{\sum_{i=1}^n \mu_i} \right)^r.$$

Lemma 2.8. (Oboudi 2019) Let $n \geq 1$ be an integer and $\rho_1 \geq \rho_2 \geq \dots \geq \rho_n$. Then

$$(\rho_1 + \dots + \rho_n)(\rho_1 + \rho_n) \geq \rho_1^2 + \dots + \rho_n^2 + n\rho_1\rho_n.$$

Lemma 2.9. (Sarasija & Binthiya 2017) For $\rho_1, \rho_2, \dots, \rho_n > 0$,

$$n \sum_{i=1}^n \rho_i - \left(\sum_{i=1}^n \sqrt{\rho_i} \right)^2 \leq n(n-1) \left(\frac{1}{n} \sum_{i=1}^n \rho_i - \left(\prod_{i=1}^n \rho_i \right)^{\frac{1}{n}} \right).$$

Lemma 2.10. (Mitrinović *et al.* 1993) If $\rho_1, \rho_2, \dots, \rho_n$ are real and $\mu_1, \dots, \mu_n$ complex numbers, then

$$\left| \sum_{k=1}^n \rho_k \mu_k \right|^2 \leq \frac{1}{2} \left(\sum_{i=1}^n |\mu_i|^2 + \left| \sum_{k=1}^n \mu_k^2 \right| \right) \left(\sum_{i=1}^n \rho_i^2 \right).$$

3. Main Results

In this section, we propose some results from $EE_{\mathcal{F}}$ of a graph.

Theorem 3.1. Consider G as a graph with a minimum dominating set of size k . If $A(L_G)B = BA(L_G)$ where $B = \begin{bmatrix} \mathbf{1}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ and $v^+ \geq k$ in which κ^+ represents the count of positive eigenvalues of $A(L_G)$, then

$$EE_F(G) = EE(G) + 2k.$$

Proof. We consider $\rho_F(G) = A(L_G) + B$ where $B = \begin{bmatrix} \mathbf{1}_k & \mathbf{0}_{k \times (m-k)} \\ \mathbf{0}_{(m-k) \times k} & \mathbf{0}_{m-k} \end{bmatrix}$, where $\mathbf{1}_i$ and $\mathbf{0}_i$ denote the identity and zero matrices of order i , respectively.

Using Lemma 2.1, since $A(L_G)B = BA(L_G)$, there are a and b of $\{1, \dots, m\}$ where for $k = 1, \dots, n$,

$$\xi_k(A_{\mathcal{F}}(G)) = \xi_{a(k)}(A(L_G)) + \xi_{b(k)}(B).$$

It is important to note that the spectrum of B includes 0 with multiplicity $m - k$ and 1 with multiplicity k . With condition $\kappa^+ \geq k$, let κ^+ represents the count of positive eigenvalues of

$A(L_G)$. Then, we have

$$\sum_{i=1}^{\kappa^+} \xi_i(A_{\mathcal{F}}(G)) = \sum_{i=1}^{\kappa^+} \xi_{a(i)}(A(L_G)) + k.$$

Using Lemma 2.2, we can get

$$\begin{aligned} EE_{\mathcal{F}}(G) &= 2 \max_{1 \leq k \leq m} \sum_{i=1}^k \xi_i(A_{\mathcal{F}}(G)) \\ &\geq 2 \sum_{i=1}^{\kappa^+} \xi_i(A(L_G)) \\ &= 2 \sum_{i=1}^{\kappa^+} \xi_{a(i)}(A(L_G)) + 2k \\ &= EE(G) + 2k. \end{aligned}$$

On the other hand, by Theorem 3.6 in Movahedi (2021), $EE_{\mathcal{F}}(G) \leq EE(G) + 2k$. Therefore, the proof is complete. $\square$

Theorem 3.2. For a graph G ,

$$EE_{\mathcal{F}}(G) \geq E(L_G).$$

Proof. According to the proof of Theorem 3.1, $A_{\mathcal{F}}(G) = A(L_G) + B$. By applying Lemma 2.3, we obtain

$$\xi_i(A_{\mathcal{F}}(G)) \geq \xi_i(A(L_G)) + \xi_m(B),$$

for $1 \leq i \leq m$ and $\xi_m(B) = 0$ is the smallest eigenvalue of B . Therefore, we get

$$2 \sum_{i=1}^{\kappa^+} \xi_i(A_{\mathcal{F}}(G)) \geq 2 \sum_{i=1}^{\kappa^+} \xi_i(A(L_G)),$$

where κ^+ represents the count of positive eigenvalues of $A_{\mathcal{F}}(G)$.

Conversely, we obtain

$$\begin{aligned} EE_{\mathcal{F}}(G) &= 2 \max_{1 \leq k \leq m} \sum_{i=1}^k \xi_i(A_{\mathcal{F}}(G)) \\ &\geq 2 \sum_{i=1}^{\kappa^+} \xi_i(A_{\mathcal{F}}(G)) \\ &\geq 2 \sum_{i=1}^{\kappa^+} \xi_i(A(L_G)) \\ &\geq 2 \sum_{i=1}^{\tau^+} \xi_i(A(L_G)), \end{aligned}$$

where $A(L_G)$ has τ^+ positive eigenvalues. Therefore, the proof is completed. $\square$

Using Lemma 2.4 and Theorem 3.2, the following result is obtained.

Corollary 3.3. *For a graph G ,*

$$E(L_G) \leq EE_{\mathcal{F}}(G) \leq E(L_G) + k.$$

Corollary 3.4. *Consider T is a tree of order n and $\mathcal{F}$ is a minimum edge dominating set with $|F| = k$. Then*

$$EE_{\mathcal{F}}(T) \leq 2(n - 2) + k.$$

Proof. Using Lemma 2.5 for the tree T , we have

$$\begin{aligned} E(L_T) &\leq 2(|V(L_T)| - 1) \\ &= 2(m - 1) \\ &= 2(n - 2). \end{aligned}$$

Therefore, by applying Corollary 3.3 and the above inequality, the result follows. $\square$

Definition 3.5. Hölder inequality (Hsu & Wang 1983). For $\rho_i, \mu_i \in \mathbb{R}$ where $1 \leq i \leq n$,

$$\sum_{i=1}^n \rho_i \mu_i \leq \left(\sum_{i=1}^n \rho_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n \mu_i^q \right)^{\frac{1}{q}}, \quad (2)$$

such that $p, q \in [1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Consider $\rho_i, \mu_i, \nu_i \in \mathbb{R}^+$ for $1 \leq i \leq n$. Then, according to Hölder inequality,

$$\left(\sum_{i=1}^n \rho_i \mu_i \nu_i \right)^3 \leq \left(\sum_{i=1}^n \rho_i^3 \right) \left(\sum_{i=1}^n \mu_i^3 \right) \left(\sum_{i=1}^n \nu_i^3 \right). \quad (3)$$

Equality holds if and only if $\rho_i = \mu_i = \nu_i$ for $i = 1, \dots, n$.

Theorem 3.6. *For a graph G with $|E(L_G)| = m'$,*

$$EE_{\mathcal{F}}(G) \geq \frac{\alpha^2}{4(D^2 + D + 2m' - t)},$$

where $\alpha = k + 2m'$, $D = \sum_{e \in \mathcal{F}} \deg(e)$ and t is the number of vertices of degree 1 in the line graph L_G .

Proof. Suppose ξ_i are the eigenvalues of $A_{\mathcal{F}}(G)$ for $1 \leq i \leq m$. By considering $\rho_i = |\xi_i|^{\frac{1}{2}}$, $\mu_i = |\xi_i|^{\frac{3}{2}}$ and $p = q = 2$ in (2), we get

$$\begin{aligned} \sum_{i=1}^m |\xi_i|^2 &= \sum_{i=1}^m |\xi_i| |\xi_i| \\ &\leq \left(\sum_{i=1}^m (|\xi_i|^{\frac{1}{2}})^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^m (|\xi_i|^{\frac{3}{2}})^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
 &= \left(\sum_{i=1}^m |\xi_i| \right)^{\frac{1}{2}} \left(\sum_{i=1}^m |\xi_i|^3 \right)^{\frac{1}{2}} \\
 &= \left(EE_{\mathcal{F}}(G) \right)^{\frac{1}{2}} \left(\sum_{i=1}^m |\xi_i|^3 \right)^{\frac{1}{2}}.
 \end{aligned}$$

Since using the relation (1), $\sum_{i=1}^m |\xi_i|^2 = k - 2m + \sum_{i=1}^n d_i^2$, we have

$$\begin{aligned}
 EE_{\mathcal{F}}(G) &\geq \frac{\left(\sum_{i=1}^m |\xi_i|^2 \right)^2}{\sum_{i=1}^m |\xi_i|^3} \\
 &\geq \frac{\alpha^2}{\sum_{i=1}^m |\xi_i|^4},
 \end{aligned} \tag{4}$$

such that $\alpha = k - 2m + \sum_{i=1}^n d_i^2$. So, it is sufficient to compute $\sum_{i=1}^m |\xi_i|^4$ in the matrix $A_{\mathcal{F}}(G)$.

We recall that $\sum_{i=1}^k (\xi_i)^k$ is the trace of $(A_{\mathcal{F}}(G))^k$. Therefore,

$$\begin{aligned}
 \sum_{i=1}^m |\xi_i|^4 &= \text{tr}(A_{\mathcal{F}})^4 = \sum_{i=1}^m \left| (A_{\mathcal{F}})_{ii}^2 \right|^2 \\
 &= \sum_{i=j} \left| (A_{\mathcal{F}})_{ij}^2 \right|^2 + \sum_{i \neq j} \left| (A_{\mathcal{F}})_{ij}^2 \right|^2 \\
 &= \alpha^2 + \sum_{i \neq j} \left| (A_{\mathcal{F}})_{ij}^2 \right|^2,
 \end{aligned} \tag{5}$$

where by applying (1), $\sum_{i=j} \left| (A_{\mathcal{F}})_{ij}^2 \right|^2 = \text{tr}((A_{\mathcal{F}})^2) = \alpha$.

For $i \neq j$, we get

$$(A_{\mathcal{F}})_{ij}^2 = \sum_{i=1}^m \omega_{ij} \omega_{ji} = \omega_{ii} \omega_{ij} + \omega_{ij} \omega_{jj} + \sum_{k \sim i, k \sim j} \omega_{ik} \omega_{kj}. \tag{6}$$

According to the definition of the matrix $A_{\mathcal{F}}(G)$, if vertices $i \sim j$ are adjacent, then $\omega_{ij} = 1$.

From the relation (6), we have $\omega_{ii} \omega_{ij} + \omega_{ij} \omega_{jj} = \omega_{ii} + \omega_{jj}$. Thus, we state the following cases:

Case 1: If $i, j \in \mathcal{F}$ in L_G , then $\omega_{ii} + \omega_{jj} = 2$.

Case 2: If only one of the vertices i and j is in $\mathcal{F}$, then $\omega_{ii} + \omega_{jj} = 1$.

Case 3: If $i, j \notin \mathcal{F}$, then $\omega_{ii} + \omega_{jj} = 0$.

Consequently, for $i \neq j$ we get

$$\sum_{i \neq j} (\omega_{ii} \omega_{ij} + \omega_{ij} \omega_{jj}) = 2 \left(\sum_{v_i \in \mathcal{F}} d(v_i) \right). \tag{7}$$

One can readily observe that for a vertex k with the degree at least 2, $\sum_{k \sim i, k \sim j} \omega_{ik} \omega_{kj} = 2$ in

Eq. (6). Assume that t and m' are the number of vertices of degree 1 and the number of edges in L_G , respectively. Since the sum of the degree of all the vertices in L_G equals $2m'$, then the number of vertices of the degree at least 2 is $2m' - t$. Therefore,

$$\sum_{i \neq j} \left(\sum_{k \sim i, k \sim j} \omega_{ik} \omega_{kj} \right) = 2(2m' - t). \quad (8)$$

Hence by applying equalities (6), (7) and (8), we have

$$\begin{aligned} \sum_{i \neq j} \left| (A_{\mathcal{F}}^2)_{ij} \right|^2 &= \sum_{i \neq j} \left| \omega_{ii} \omega_{ij} + \omega_{ij} \omega_{jj} + \sum_{k \sim i, k \sim j} \omega_{ik} \omega_{kj} \right|^2 \\ &= \sum_{i \neq j} (\omega_{ii} \omega_{ij} + \omega_{ij} \omega_{jj})^2 + \sum_{i \neq j} \left(\sum_{k \sim i, k \sim j} \omega_{ik} \omega_{kj} \right)^2 \\ &\quad + 2 \sum_{i \neq j} (\omega_{ii} \omega_{ij} + \omega_{ij} \omega_{jj}) \left(\sum_{k \sim i, k \sim j} \omega_{ik} \omega_{kj} \right) \\ &= 4 \left(\sum_{e \in \mathcal{F}} \deg(e) \right)^2 + 4(2m' - t) + 4 \left(\sum_{e \in \mathcal{F}} \deg(e) \right). \end{aligned}$$

By putting $D = \sum_{e \in \mathcal{F}} \deg(e)$, we get

$$\sum_{i \neq j} \left| (A_{\mathcal{F}}^2)_{ij} \right|^2 = 4(D^2 + D + 2m' - t). \quad (9)$$

By substituting Eq. (9) in inequality (4), the result follows. $\square$

Theorem 3.7. For a graph G ,

$$EE_{\mathcal{F}}(G) \leq \sqrt[3]{m^2} \sqrt{\alpha},$$

where $\alpha = k - 2m + \sum_{i=1}^n d_i^2$.

Proof. Suppose ξ_i are the eigenvalues of $A_{\mathcal{F}}(G)$ for $1 \leq i \leq m$. We recall for non-negative x_i for $1 \leq i \leq n$ and $k \geq 2$.

$$\sum_{i=1}^n (x_i)^k \leq \left(\sum_{i=1}^n x_i^n \right)^{\frac{k}{n}}. \quad (10)$$

By considering $\rho_i = |\xi_i|$, $\mu_i = c_i = 1$ for $i = 1, \dots, n$ in Eq. (3) and by applying Eq. (10), we get

$$\begin{aligned} \left(\sum_{i=1}^m |\xi_i| \right)^3 &\leq \left[\sum_{i=1}^m |\xi_i|^3 \right] \left[\sum_{i=1}^m (1)^3 \right] \left[\sum_{i=1}^m (1)^3 \right] \\ &\leq m^2 \left[\sum_{i=1}^m |\xi_i|^2 \right]^{\frac{3}{2}}. \end{aligned}$$

By applying the relation (1), we have

$$\left(EE_{\mathcal{F}}(G) \right)^3 \leq m^2 [\alpha]^{\frac{3}{2}}.$$

Therefore,

$$EE_{\mathcal{F}}(G) \leq m^{\frac{2}{3}} [\alpha]^{\frac{1}{2}}. \quad \square$$

Definition 3.8. Cauchy-Schwarz inequality (Aldaz *et al.* 2015). For $\rho_i, \mu_i \in \mathbb{R}$ where $1 \leq i \leq n$,

$$\left(\sum_{i=1}^n \rho_i \mu_i \right)^2 \leq \left(\sum_{i=1}^n \rho_i^2 \right) \left(\sum_{i=1}^n \mu_i^2 \right).$$

Equality holds if and only if $\rho_i = \zeta \mu_i$ for a non-zero constant $\zeta \in \mathbb{R}$.

Theorem 3.9. Let G be a simple graph and $\alpha' = \alpha(L_G)$ be the independence number of line graph L_G . Then

$$EE_{\mathcal{F}}(G) \leq \sqrt{2\alpha(m - \alpha')},$$

where $\alpha = k - 2m + \sum_{i=1}^n d_i^2$.

Proof. Suppose ξ_i are the eigenvalues of $A_{\mathcal{F}}(G)$ for $1 \leq i \leq m$ such that $A_{\mathcal{F}}(G)$ has κ^+ and κ^- positive and negative eigenvalues, respectively. Therefore, L_G has $m - \kappa^+ - \kappa^-$ eigenvalues which are equal to zero. Using Lemma 2.6, we have

$$\alpha' = \alpha(L_G) \leq (m - \kappa^+ - \kappa^-) + \min\{\kappa^+, v^-\}.$$

Therefore, $\alpha' \leq (m - \kappa^+ - \kappa^-) + \kappa^+$ and $\alpha' \leq (m - \kappa^+ - \kappa^-) + \kappa^-$. Consequently, $v^+ \leq m - \alpha'$ and $v^- \leq m - \alpha'$. Using the relation (1), we can get

$$\begin{aligned} EE_{\mathcal{F}}(G) &= \sum_{i=1}^m |\xi_i| \\ &= \sum_{i=1}^{\kappa^+} \xi_i - \sum_{i=1}^{\kappa^-} \xi_i \\ &= 2 \sum_{i=1}^{\kappa^+} \xi_i - \sum_{i=1}^m \xi_i \\ &= 2 \sum_{i=1}^{\kappa^+} \xi_i - k. \end{aligned}$$

Thus, $EE_{\mathcal{F}}(G) + k = 2 \sum_{i=1}^{\kappa^+} \xi_i$.

We have

$$\begin{aligned} EE_{\mathcal{F}}(G) &= \sum_{i=1}^{\kappa^+} \xi_i - \sum_{i=1}^{\kappa^-} \xi_i \\ &= \sum_{i=1}^{\kappa^+} \xi_i + \sum_{i=1}^{\kappa^-} |\xi_i| \end{aligned}$$

$$= 2 \sum_{i=1}^{\kappa^-} \xi_i + k.$$

Thus, $EE_{\mathcal{F}}(G) - k = 2 \sum_{i=1}^{\kappa^-} |\xi_i|$.

From Cauchy-Schwarz inequality, we can get

$$EE_{\mathcal{F}}(G) + k = 2 \sum_{i=1}^{\kappa^+} \xi_i \leq 2 \sqrt{\kappa^+ \sum_{i=1}^{\kappa^+} \xi_i^2}. \quad (11)$$

Similarity,

$$EE_{\mathcal{F}}(G) - k = 2 \sum_{i=1}^{\kappa^-} |\xi_i| \leq 2 \sqrt{\kappa^- \sum_{i=1}^{\kappa^-} \xi_i^2}. \quad (12)$$

Using Cauchy-Schwarz inequality and inequalities (11) and (12), we get

$$\begin{aligned} (EE_{\mathcal{F}}(G))^2 &= \left(\frac{EE_{\mathcal{F}}(G) + k}{2} + \frac{EE_{\mathcal{F}}(G) - k}{2} \right)^2 \\ &\leq 2 \left[\left(\frac{EE_{\mathcal{F}}(G) + k}{2} \right)^2 + \left(\frac{EE_{\mathcal{F}}(G) - k}{2} \right)^2 \right] \\ &\leq 2 \left[\kappa^+ \sum_{i=1}^{\kappa^+} \xi_i^2 + \kappa^- \sum_{i=1}^{\kappa^-} \xi_i^2 \right] \\ &\leq 2 \left[(m - \alpha') \sum_{i=1}^{\kappa^+} \xi_i^2 + (m - \alpha') \sum_{i=1}^{\kappa^-} \xi_i^2 \right] \\ &= 2 \left[(m - \alpha') \left(\sum_{i=1}^{\kappa^+} \xi_i^2 + \sum_{i=1}^{\kappa^-} \xi_i^2 \right) \right] \\ &= 2 \left[(m - \alpha') \left(\sum_{i=1}^m \xi_i^2 \right) \right] \\ &= 2[(m - \alpha')\alpha]. \end{aligned}$$

where using the relation (1), $\sum_{i=1}^m \xi_i^2 = k - 2m + \sum_{i=1}^n d_i^2 = \alpha$. Thus,

$$EE_{\mathcal{F}}(G) \leq \sqrt{2[(m - \alpha')\alpha]}. \quad \square$$

Theorem 3.10. Consider G as a simple graph. Then

$$EE_{\mathcal{F}}(G) \leq \sqrt[3]{mk\alpha},$$

where $\alpha = k - 2m + \sum_{i=1}^n d_i^2$.

Proof. Suppose ξ_i are the eigenvalues of $A_{\mathcal{F}}(G)$ for $1 \leq i \leq m$. Applying Lemma 2.7(i) and

putting $\rho_i = |\xi_i|$, $\mu_i = |\xi_i|^2$ and $r = \frac{1}{2}$, we get

$$\sum_{i=1}^m \frac{|\xi_i|^{\frac{3}{2}}}{|\xi_i|} \geq \frac{\left(\sum_{i=1}^m |\xi_i|\right)^{\frac{3}{2}}}{\left(\sum_{i=1}^m |\xi_i|^2\right)^{\frac{1}{2}}}.$$

Using (1), in the above relation we consider $\sum_{i=1}^m |\xi_i| = EE_{\mathcal{F}}(G)$ and $\sum_{i=1}^m |\xi_i|^2 = \alpha$ in which $\alpha = k - 2m + \sum_{i=1}^n d_i^2$. Therefore, we obtain

$$(EE_{\mathcal{F}}(G))^{\frac{3}{2}} \leq \alpha^{\frac{1}{2}} \sum_{i=1}^m |\xi_i|^{\frac{1}{2}}.$$

Thus,

$$(EE_{\mathcal{F}}(G))^3 \leq \alpha \left(\sum_{i=1}^m |\xi_i|^{\frac{1}{2}} \right)^2.$$

By applying the Cauchy-Schwarz inequality and setting $\rho_i = 1$ and $\mu_i = |\xi_i|^{\frac{1}{2}}$ on the right-hand side of the given relation, we obtain

$$\begin{aligned} (EE_{\mathcal{F}}(G))^3 &\leq \alpha \left(\sum_{i=1}^m |\xi_i| \right) \left(\sum_{i=1}^m 1 \right) \\ &= m\alpha(|\mathcal{F}|) \\ &= mk\alpha. \end{aligned}$$

Hence the result follows easily. $\square$

Theorem 3.11. Consider G as a simple graph with $|\xi_1| \geq \dots \geq |\xi_m|$. Then

$$EE_{\mathcal{F}}(G) \geq \frac{\alpha + m|\xi_1||\xi_m|}{|\xi_1| + |\xi_m|},$$

where $\alpha = k - 2m + \sum_{i=1}^n d_i^2$.

Proof. Consider $|\xi_1| \geq |\xi_2| \geq \dots \geq |\xi_m|$ the nonincreasing order of the absolute values of eigenvalues of $A_{\mathcal{F}}(G)$.

According to Lemma 2.8 and putting $\rho_i = |\xi_i|$ for $i = 1, \dots, m$, we get

$$\left(|\xi_1| + \dots + |\xi_m|\right) \left(|\xi_1| + |\xi_m|\right) \geq |\xi_1|^2 + \dots + |\xi_m|^2 + m|\xi_1||\xi_m|.$$

Since $\sum_{i=1}^m |\xi_i| = EE_{\mathcal{F}}(G)$ and by relation (1), we have

$$(EE_{\mathcal{F}}(G)) \left(|\xi_1| + |\xi_m|\right) \geq \alpha + m|\xi_1||\xi_m|,$$

where $\alpha = k - 2m + \sum_{i=1}^n d_i^2$. Hence,

$$EE_{\mathcal{F}}(G) \geq \frac{\alpha + m|\xi_1||\xi_m|}{|\xi_1| + |\xi_m|}.$$

The proof is completed. $\square$

Theorem 3.12. Consider G as a simple graph. Then

$$EE_{\mathcal{F}}(G) \geq mP^{\frac{1}{m}},$$

where $P = \det(A_{\mathcal{F}}(G))$.

Proof. Suppose that ξ_i are the eigenvalues of $A_{\mathcal{F}}(G)$ for $1 \leq i \leq m$. Applying Lemma 2.9 and putting $\rho_i = |\xi_i|$, we get

$$m \sum_{i=1}^m |\xi_i| - \left(\sum_{i=1}^m |\xi_i|^{\frac{1}{2}} \right)^2 \leq m(m-1) \left(\frac{1}{m} \sum_{i=1}^m |\xi_i| - \left(\prod_{i=1}^m |\xi_i| \right)^{\frac{1}{m}} \right).$$

Since $\prod_{i=1}^m |\xi_i| = \det(A_{\mathcal{F}}(G))$ and using the definitions, we have

$$mEE_{\mathcal{F}}(G) - \left(\sum_{i=1}^m |\xi_i|^{\frac{1}{2}} \right)^2 \leq (m-1)EE_{\mathcal{F}}(G) - m(m-1)P^{\frac{1}{m}},$$

where $P = \det(A_{\mathcal{F}}(G))$. Therefore, we can have

$$EE_{\mathcal{F}}(G) \leq \left(\sum_{i=1}^m |\xi_i|^{\frac{1}{2}} \right)^2 - m(m-1)P^{\frac{1}{m}}.$$

With considering the Cauchy-Schwarz inequality and setting $\rho_i = |\xi_i|^{\frac{1}{2}}$ and $\mu_i = 1$ on the right side of the above relation, we obtain

$$\begin{aligned} EEE_{\mathcal{F}}(G) &\leq \left(\sum_{i=1}^m |\xi_i|^{\frac{1}{2}} \right)^2 - m(m-1)P^{\frac{1}{m}} \\ &\leq \left(\sum_{i=1}^m |\xi_i| \right) \left(\sum_{i=1}^m 1 \right) - m(m-1)P^{\frac{1}{m}} \\ &= mEE_{\mathcal{F}}(G) - m(m-1)P^{\frac{1}{m}}. \end{aligned}$$

Hence,

$$m(m-1)P^{\frac{1}{m}} \leq (m-1)EE_{\mathcal{F}}(G).$$

Since $m \geq 1$, we have

$$EE_{\mathcal{F}}(G) \geq mP^{\frac{1}{m}}. \quad \square$$

Theorem 3.13. Consider G as a simple graph. Then

$$EE_{\mathcal{F}}(G) \leq k - 2m + M(G),$$

where $M(G)$ is the Zagreb index of graph G .

Proof. Suppose ξ_i are the eigenvalues of $A_{\mathcal{F}}(G)$ for $1 \leq i \leq m$. With considering $\rho_i = |\xi_i|^{\frac{1}{2}}$,

$\mu_i = |\xi_i|^{\frac{1}{2}}$ and $r = 2$ in Lemma 2.7(ii), we get

$$\sum_{i=1}^m |\xi_i|^{\frac{1}{2}} \left(|\xi_i|^{\frac{1}{2}} \right)^2 \geq \sum_{i=1}^m |\xi_i|^{\frac{1}{2}} \left(\frac{\sum_{i=1}^m |\xi_i|^{\frac{1}{2}} |\xi_i|^{\frac{1}{2}}}{|\xi_i|^{\frac{1}{2}}} \right)^2.$$

Thus, we get

$$\sum_{i=1}^m |\xi_i|^{\frac{3}{2}} \geq \frac{\left(\sum_{i=1}^m |\xi_i| \right)^2}{\sum_{i=1}^m |\xi_i|^{\frac{1}{2}}}.$$

Therefore using the relation (1) and the definitions, we have

$$\begin{aligned} \left(EE_{\mathcal{F}}(G) \right)^2 &\leq \left(\sum_{i=1}^m |\xi_i|^{\frac{1}{2}} \right) \left(\sum_{i=1}^m |\xi_i|^{\frac{3}{2}} \right) \\ &\leq \left(\sum_{i=1}^m |\xi_i| \right) \left(\sum_{i=1}^m |\xi_i|^2 \right) \\ &= EE_{\mathcal{F}}(G) \left(k - 2m + \sum_{i=1}^n d_i^2 \right). \end{aligned}$$

Since $EE_{\mathcal{F}}(G) > 0$ and the Zagreb index of graph G is equal to $M(G) = \sum_{i=1}^n d_i^2$, then

$$EE_{\mathcal{F}}(G) \leq k - 2m + M(G). \quad \square$$

We utilize the following inequality, which will be essential for proving the next result.

Lemma 3.14. Mitrinović *et al.* (1993) *Let $x_k \in (0, 1)$, $\rho_k \geq 1$ for $k = 1, \dots, n$. Then,*

$$\prod_{k=1}^n (1 + x_k)^{\rho_k} > 1 + \sum_{k=1}^n \rho_k x_k.$$

Theorem 3.15. *Consider G as a simple graph with $|\xi_1| \geq \dots \geq |\xi_m| > 0$. Then*

$$EE_{\mathcal{F}}(G) < 2^{m-1} |\xi_1|.$$

Proof. Consider $|\xi_1| \geq |\xi_2| \geq \dots \geq |\xi_m|$ the nonincreasing order of the absolute values of eigenvalues of $A_{\mathcal{F}}(G)$. With putting $x_k = \frac{|\xi_k|}{|\xi_1|}$ for $2 \leq k \leq m$ and $\rho_k = 1$ in Lemma 3.14, we get

$$\begin{aligned} \prod_{k=1}^n (1 + x_k)^{\rho_k} &= \prod_{k=2}^m \left(1 + \frac{|\xi_k|}{|\xi_1|} \right) \\ &> 1 + \sum_{k=2}^m \frac{|\xi_k|}{|\xi_1|} \\ &= 1 + \frac{1}{|\xi_1|} \sum_{k=2}^m |\xi_k| \end{aligned}$$

$$\begin{aligned}
 &= 1 + \frac{1}{|\xi_1|} \left(EE_{\mathcal{F}}(G) - |\xi_1| \right) \\
 &= 1 + \frac{1}{|\xi_1|} EE_{\mathcal{F}}(G) - 1 \\
 &= \frac{1}{|\xi_1|} EE_{\mathcal{F}}(G).
 \end{aligned}$$

Therefore, we get

$$\begin{aligned}
 EE_{\mathcal{F}}(G) &< |\xi_1| \left(\frac{\prod_{k=2}^m (|\xi_1| + |\xi_k|)}{|\xi_1|^{m-1}} \right) \\
 &= \frac{1}{|\xi_1|^{m-2}} \left(\prod_{k=2}^m (|\xi_1| + |\xi_k|) \right) \\
 &= \frac{1}{|\xi_1|^{m-2}} \left(\prod_{k=2}^m (|\xi_1| + |\xi_1|) \right) \\
 &= \frac{1}{|\xi_1|^{m-2}} \left(\prod_{k=2}^m (2|\xi_1|) \right) \\
 &= \frac{1}{|\xi_1|^{m-2}} (2|\xi_1|)^{m-1} \\
 &= 2^{m-1} |\xi_1|,
 \end{aligned}$$

and this leads to the desired bound. $\square$

Theorem 3.16. For a simple graph G ,

$$EE_{\mathcal{F}}(G) \geq k,$$

the equality holds if and only if $A_{\mathcal{F}}(G)$ is the positive semidefinite matrix.

Proof. Suppose ξ_i for $1 \leq i \leq m$ are the eigenvalues of $A_{\mathcal{F}}(G)$. According to Lemma 2.10 and putting $\rho_k = \xi_k^{\frac{1}{2}}$ and $z_k = \xi_k^{\frac{1}{2}}$ for $1 \leq k \leq m$, we have

$$\begin{aligned}
 k^2 &= \left| \sum_{k=1}^m \xi_k \right|^2 = \left| \sum_{k=1}^m \xi_k^{\frac{1}{2}} \right|^2 \\
 &\leq \frac{1}{2} \left(\sum_{k=1}^m |\xi_k^{\frac{1}{2}}|^2 + \left| \sum_{k=1}^m (\xi_k^{\frac{1}{2}})^2 \right| \right) \left(\sum_{k=1}^m (\xi_k^{\frac{1}{2}})^2 \right) \\
 &= \frac{1}{2} (EE_{\mathcal{F}}(G) + k) k.
 \end{aligned}$$

Therefore,

$$k \leq \frac{EE_{\mathcal{F}}(G)}{2} + \frac{k}{2},$$

and consequently, we get $EE_{\mathcal{F}}(G) \geq k$.

The equality holds if and only if $EE_{\mathcal{F}}(G) = \sum_{i=1}^m |\xi_i| = k$. On the other hand, $\sum_{i=1}^m \xi_i = k$. Therefore, if $A_{\mathcal{F}}(G)$ is the positive semidefinite matrix, that is $|\xi_i| = \xi_i$, then $EE_{\mathcal{F}}(G) = k$.

□

4. Conclusions

The minimum edge dominating energy combines the idea of edge domination with spectral graph theory (the study of graph properties through eigenvalues). It aims to quantify a certain form of "energy" related to the edge domination properties of a graph. Research in this area focuses on establishing relationships between this energy and other graph energies and bound properties. In this paper, we obtained new bounds for $EE_{\mathcal{F}}$ of graphs as indicated in theorems and lemmas in Section 3. Researchers are also analyzing $EE_{\mathcal{F}}$ of Cayley graphs to understand the structural properties of the underlying groups. Similar analysis can be performed on graphs that are produced from the structure of mathematical rings. The next future research directions are maybe to work on $EE_{\mathcal{F}}$ for groups and rings.

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*Department of Mathematics, Faculty of Sciences
Golestan University
Gorgan, IRAN
E-mail: f.movahedi@gu.ac.ir*

*Department of Mathematics
Estahban Branch, Islamic Azad University
Estahban, IRAN
E-mail: hadi.akhbari@iau.ac.ir*

*Special Interest Group of Modeling and Data Analytics (SIGMDA)
Faculty of Computer Science and Mathematics
Universiti Malaysia Terengganu
21030 Kuala Nerus
Terengganu, MALAYSIA
E-mail: hroslan@umt.edu.my**

*Department of Mathematics
College of Computer Science and Mathematics
Tikrit University
34001 Tikrit, IRAQ
E-mail: nabarif@tu.edu.iq*

*Faculty of Computer Science and Mathematics
UiTM Shah Alam
40450 Shah Alam
Selangor, MALAYSIA
E-mail: sumarni164@uitm.edu.my*

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*Corresponding author