

NEWTON ITERATION WITH WEIGHTED EXPONENTIAL PENALTY FUNCTION FOR SOLVING MULTI-OBJECTIVE CONSTRAINED OPTIMIZATION PROBLEMS

*(Lelaran Newton dengan Fungsi Penalti Eksponen Berpemberat bagi Menyelesaikan Masalah
Pengoptimuman Kekangan Pelbagai Objektif)*

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ABSTRACT

In this paper, the Newton iterative method and weighted exponential penalty function method for solving multi-objective constrained optimization problems are proposed on the basis of classical optimization methods for mathematical planning. Firstly, according to the characteristics of the objective function, it is transformed into a single-objective constrained optimization problem using variable weight coefficients. Then according to the characteristics of the constraints, the exponential penalty function method is used to transform it into an unconstrained optimization problem. Finally, Newton's method is used to solve the transformed unconstrained optimization problem to obtain the efficient Pareto solution of the original problem. The convergence of the method is included in the paper, and numerical experiments show that our proposed method can obtain a set of effective Pareto solutions corresponding to multi-objective optimization problems in different dimensions.

Keywords: multi-objective constrained optimization; weighting method; exponential penalty function method; Newton's method

ABSTRAK

Dalam makalah ini, kaedah lelaran Newton dan kaedah fungsi penalti eksponen berpemberat bagi menyelesaikan masalah pengoptimuman kekangan pelbagai objektif yang dipertimbangkan berdasarkan kepada kaedah pengoptimuman klasik terhadap perancangan matematik. Berpanduan kepada ciri-ciri fungsi objektif, permasalahan tersebut perlu diolahkan kepada masalah pengoptimuman kekangan objektif tunggal menggunakan pekali pemberat berubah-ubah. Seterusnya berasaskan kepada ciri-ciri kekangan, kaedah fungsi penalti eksponen digunakan untuk mengolahnya kepada masalah pengoptimuman tanpa kekangan. Akhir sekali, kaedah Newton dilaksanakan untuk menyelesaikan masalah pengoptimuman tanpa kekangan yang dijanakan bagi mendapatkan penyelesaian Pareto yang efisien bagi permasalahan asal. Penumpuan kaedah tersebut juga dimuat dalam makalah ini dan ujikaji berangka menunjukkan bahawa kaedah dicadangkan tersebut boleh memperoleh satu set penyelesaian Pareto yang efektif yang sepadan dengan masalah pengoptimuman pelbagai objektif dalam dimensi yang berbeza.

Kata kunci: pengoptimuman kekangan pelbagai objektif; kaedah pemberat; kaedah fungsi penalti eksponen; kaedah Newton

1. Introduction

Multi-objective optimization problems are widely used in the fields of production scheduling (Ojstersek *et al.* 2020), water resource allocation (Naghdi *et al.* 2021) and product design (Zhou *et al.* 2021), for example, in production scheduling, it is necessary to reduce switching by

placing the right work order on the right production line, but also to ensure that the line is balanced, and it is necessary to maximize the production capacity but also to ensure that all the work orders received cannot be overdue. In water resource allocation, it is necessary to ensure that the regional water supply is sufficient and the water resource utilisation rate is high, but also to ensure that the production cost of the water supply system is low and to achieve a certain degree of economic efficiency. In production design, we want to design a new product that has both a high user experience and a low production cost. Abstracting these problems into mathematical models gives rise to multi-objective optimization problems. Solving multi-objective optimization problems is challenging because these multiple objectives are conflicting and most objective functions have one or more constraints. Unlike single-objective optimization problems, multi-objective problems have a wide variety of constraints of type, where the optimization of one objective result can lead to non-optimization or even performance degradation of one or more other objectives, and therefore require well-designed algorithms to obtain well-distributed Pareto-optimal or non-dominated solutions. The main multi-objective constrained optimization problem solved in this paper can be expressed as follows:

$$\begin{aligned} \min F(x) &= (f_1(x), f_2(x), \dots, f_m(x))^T, \\ \text{s.t. } g_j(x) &\leq 0, j = 1, \dots, p, \\ x &\in \Omega, \end{aligned} \tag{1}$$

where $x = (x_1, x_2, \dots, x_n)^T \in R^n$ is the n – dimensional decision vector, $\Omega = [x_i^L, x_i^U]^n \subseteq R^n$ denotes the decision space, $F(x) = (f_1(x), f_2(x), \dots, f_m(x))^T \in R^m$ is the objective function and R^m is the m -dimensional objective space, $g_j(x) \leq 0$ is the j -th inequality constraint, we refer to the solved problem (1) as a multi-objective optimization problem (*MOOP*), and let X denote the set of all feasible solutions to the problem (*MOOP*), i.e., $X = \{x \in R^n: g_j(x) \leq 0, j = 1, \dots, p\}$.

The main methods for the solution of problem (*MOOP*) include classical optimization methods and intelligent optimization algorithms. Classical methods include the linear weighted method, the main objective method, and the ideal point method (Wang *et al.* 2016), evaluation function method (Gunantara 2018). Intelligent optimization algorithms mainly include Genetic Algorithms (GA) (Konak *et al.* 2006), Simulated Annealing Algorithm (SA) (Cakir *et al.* 2011), Ant Colony Algorithm (ACA) (Ning *et al.* 2019), Particle Swarm Algorithm (PSO) (Fan & Chang 2009) and so on. Intelligent optimization algorithms are a class of methods that mimic the mechanism of biological evolution in nature to produce a global probabilistic search, in which the function is not required to have the properties of continuity, differentiability and non-convexity, and a group of mutually non-dominated solutions can be obtained. The disadvantage of this method is that problem (1) needs to be decomposed and then solved one by one, and the difficulty of fitness assignment and diversity control increases with the increase of dimension. Secondly, because it is a stock-based algorithm, the running time is long and susceptible to the influence of the setting of parameters. Classical multi-objective optimization methods have more obvious advantages, such as the possibility of exploiting the function's differentiability, a sound theory, and a small amount of computation, fast convergence and an obvious termination criterion. Therefore, in this article, we primarily consider the linear weighting method to solve problem (1).

Thus, the problem (*MOOP*) is converted into a solution about a single objective constrained optimization problem. The exponential penalty function method is one class of exterior point penalty function, using which a constrained optimization problem can be efficiently converted to an unconstrained optimization problem. The method was first proposed by Murphy (Murphy

1974), then Cominetti showed that the exponential penalty function (EPF) method can effectively convert a constrained optimization problem into an unconstrained optimization problem and pointed out that the EPF method is analogous to the interior penalty function method, with super-linear convergence (Cominetti & Dussault 1994). Jayswal applied the EPF method to fractional multi-objective constrained optimization problems and discussed the convergence of the method (Jayswal & Choudhury 2014). Recently, Jayswal also applied the method to multidimensional variational problems with inequality constraints and discussed convergence, concluding that the sequence of the converging subsequence of the minimum value of the unconstrained multidimensional variational problem converges to the optimal solution of the multidimensional problem (Jayswal & Baranwal 2023). Based on these research experiences, we consider combining the method of EPF and the weighting method to convert the multi-objective constrained optimization problem.

The methods for solving unconstrained optimization problems include: the fastest descent method (Nocedal 1992), the Newton method (Dennis & Schnabel 1996), the improved Newton method (Ghazali *et al.* 2020a; Cheng *et al.* 2025), among them, the Newton method is the classical method to solve the unconstrained optimization problems. The Newton iteration direction is constructed under the requirement that both the objective function and the constraint function are second-order continuously differentiable, so that the orientation of each objective function is in the decreasing direction under the feasible condition, and then the optimal solution is finally obtained through iteration. Therefore, it is very widely used, and this paper mainly considers the Newton method for solving the transformed unconstrained optimization problem.

Therefore on the basis of the research of Jayswal and Baranwal (2023) we publish this paper, which next consists of the following main sections, the second section introduces some preparatory knowledge about ordinal relations and multi-objective solution sets, the third section describes the weighted EPF method used in this paper and the proof of convergence, the fourth section gives the algorithms is given to solve the converted unconstrained optimization problem, the fifth section gives the numerical experiments and presentation of the results. Lastly, some of the conclusions are discussed in the sixth section.

2. Notations and Preliminaries

In this section, we will state some basic concepts and fundamental theorems which will lay the foundation for the solution methods of multi-objective constrained optimization problems introduced later. Firstly, we introduce the comparison relation between vectors in a vector space, or the relation called 'order', and we give a basic relation for the order of vectors below.

Definition 2.1. Let $a = (a_1, a_2, \dots, a_m)^T$, $b = (b_1, b_2, \dots, b_m)^T$ be two vectors in an m -dimensional Euclidean space R^m .

- (1) If $a_i = b_i (i = 1, \dots, m)$, then vector a is said to be equal to vector b , denoted $a = b$.
- (2) If $a_i \leq b_i (i = 1, \dots, m)$, then the vector a is said to be less than or equal to the vector b , denoted as $a \leq b$ or $b \geq a$.
- (3) If $a_i \leq b_i (i = 1, \dots, m)$ and at least one of them is a strict inequality, then vector a is said to be less than a vector b , denoted $a < b$ or $b > a$.
- (4) If $a_i < b_i (i = 1, \dots, m)$, then vector a is said to be strictly smaller than vector b , denoted as $a < b$ or $b > a$.

The order between vectors determined by the above definition is called the natural order of the vectors, and in particular, when $m = 1$, the above natural order is identical with that of the real order, except that then $<$ and $<$ have the same meaning.

A few basic concepts of the general multi-objective model (1) are given below using the natural order of vectors.

Definition 2.2. $X \subseteq R^n$ is a feasible set of solutions of the model (MOOP) and $f(x) \in R^m$ is a vector objective function, which is said to be a valid solution of the problem (MOOP) if $x^* \in X$ and there exists no $x \in X$ such that $f(x) < f(x^*)$, which is said to be a valid solution to the problem (MOOP), and is said to be a weakly valid solution to the problem (MOOP) if $x^* \in X$ and there exists no $x \in X$ such that $f(x) < f(x^*)$, then it is said to be a weakly efficient solution of the (MOOP).

We define the set of weakly effective solutions of (MOOP) as $\hat{X}$.

If A, B are two subsets of R^n , then $x \in A \setminus B$ means $x \in A$ but $x \notin B$.

Let $A_k \subset R^n, k \in N = \{1, 2, \dots\}$, then we denote

$$\overline{\lim}_{k \rightarrow \infty} A_k = \{x \in R^n : x \in A_k \text{ for infinitely many } k \in N\};$$

$$\underline{\lim}_{k \rightarrow \infty} A_k = \{x \in R^n : x \in A_k \text{ for all but finitely many } k \in N\};$$

it is clear that $\underline{\lim}_{k \rightarrow \infty} A_k \subseteq \overline{\lim}_{k \rightarrow \infty} A_k$, if $\overline{\lim}_{k \rightarrow \infty} A_k \subseteq \underline{\lim}_{k \rightarrow \infty} A_k$ then there is

$$\lim_{k \rightarrow \infty} A_k = \overline{\lim}_{k \rightarrow \infty} A_k = \underline{\lim}_{k \rightarrow \infty} A_k.$$

3. Weighting Exponential Penalty Function Method and Convergence

3.1. Weighting method

We consider a linear weighting approach to convert the problem (MOOP) to a single-objective constrained optimization problem with the expression below:

$$\begin{aligned} \min F(x) &= w_1 f_1(x) + w_2 f_2(x) + \dots + w_m f_m(x) \\ \text{s.t. } g_j(x) &\leq 0, j = 1, \dots, p, \\ x &\in R^n. \end{aligned} \tag{2}$$

where $w_i (i = 1, \dots, m)$ is the weighting factor, $\sum_{i=1}^m w_i = 1$. We define problem (2) as $(MOOP)_w$.

Definition 3.1. It is known that $X \subseteq R^n$ is a feasible solution set for the model (MOOP), and obviously for $(MOOP)_w$, if $x^* \in X$ and there exists no $x \in X$ such that

$$F(x) \leq F(x^*).$$

Then $x^* \in X$ is said to be the optimal solution of the constrained optimization problem $(MOOP)_w$.

The following lemma establishes the equivalence between the solution of the linearly weighted problem $(MOOP)_w$ and the original multi-objective optimization problem (MOOP).

Lemma 3.1. (Miettinen 1999) *The optimal solution of the linear weighted problem $(MOOP)_w$ is a weak pareto optimal solution of the problem (MOOP).*

Lemma 3.2. (Miettinen 1999) Let the problem (MOOP) be convex, if x^* is its pareto optimal solution, then there exists a weight vector $w(w_i > 0, i = 1, \dots, m, \sum_{i=1}^m w_i = 1)$ such that x^* is the optimal solution of the weighted optimization problem (MOOP) $_w$.

3.2. Exponential penalty function method

In the following, we transform (MOOP) into the following unconstrained optimization problem by introducing a new variable ρ and using the EPF method to combine the inequality constraints with the objective function $F(x)$, i.e.

$$L(x, \rho_k) = F(x) + \frac{1}{\rho_k} \sum_{j=1}^p \phi[\rho_k g_j(x)], \quad (3)$$

where, $F(x) = w_1 f_1(x) + w_2 f_2(x) + \dots + w_m f_m(x)$, $\rho_k > 0$, and $\lim_{k \rightarrow \infty} \rho_k = +\infty$, $\phi(t) = e^{t-1}$. We define the problem (3) as (MOOP) $_{w\rho}$.

Definition 3.2. (Jayswal & Baranwal 2023) If $x^* \in X$ and there exists no $x \in X$ such that

$$L(x, \rho_k) \leq L(x^*, \rho_k).$$

Then $x^* \in X$ is said to be an optimal solution to the constrained optimization problem (MOOP) $_{w\rho}$.

We define the set of optimal solutions of (MOOP) $_{w\rho}$ as $\widehat{X}_k$.

Next, we attempt to establish several lemmas and theorems closely related to the relation between the feasible solutions of the (MOOP) problem and the limit point of an EPF with penalty parameter ρ_k . Actually, the proof ideas of Lemmas 3.3 and 3.4 and Theorems 3.1-3.6 have been motivated by the works of Liu and Feng (Liu & Feng 2010), Jayswal and Choudhury (Jayswal & Choudhury 2014), and Jayswal and Baranwal (Jayswal & Baranwal 2023). The following lemmas are crucial because some important results of this paper need to be proved by them.

Lemma 3.3. Let X be a feasible solution to the original problem (MOOP)

(1) if $x \in X$, then

$$\lim_{\rho \rightarrow +\infty} \frac{1}{\rho} \sum_{j=1}^p \phi[\rho g_j(x)] = 0.$$

(2) if $x \notin X$, then

$$\lim_{\rho \rightarrow +\infty} \frac{1}{\rho} \sum_{j=1}^p \phi[\rho g_j(x)] = +\infty.$$

Proof. (1) Let $x \in X$, then $g_j(x) \leq 0, j = 1, \dots, p$. If $g_j(x) = 0$, an arbitrary $j = 1, \dots, p$, it is clear that the following result follows

$$\lim_{\rho \rightarrow +\infty} \frac{1}{\rho} \sum_{j=1}^p \phi[\rho g_j(x)] = \lim_{\rho \rightarrow +\infty} \frac{1}{\rho} \sum_{j=1}^p e^{[\rho g_j(x)-1]} = 0.$$

If $g_j(x) < 0$, then $\lim_{\rho \rightarrow +\infty} e^{\rho g_j(x)} = 0$,

$$\begin{aligned} \lim_{\rho \rightarrow +\infty} \frac{1}{\rho} \sum_{j=1}^p \phi[\rho g_j(x)] &= \lim_{\rho \rightarrow +\infty} \frac{1}{\rho} \sum_{j=1}^p e^{[\rho g_j(x)-1]} = \lim_{\rho \rightarrow +\infty} \sum_{j=1}^p \frac{e^{\rho g_j(x)}}{\rho e} \\ &= \sum_{j=1}^p \frac{\lim_{\rho \rightarrow +\infty} e^{\rho g_j(x)}}{\lim_{\rho \rightarrow +\infty} \rho e} = 0. \end{aligned}$$

(2) if $x \notin X$, then $g_j(x) > 0$, for some $j \in \{1, \dots, p\}$, let $g_{j_1}(x) > 0$, where $j_1 \in \{1, \dots, p\}$, then

$$\begin{aligned} \lim_{\rho \rightarrow +\infty} \frac{1}{\rho} \sum_{j=1}^p \phi[\rho g_j(x)] &= \lim_{\rho \rightarrow +\infty} \frac{1}{\rho} \left\{ \sum_{\substack{j=1 \\ j \neq j_1}}^p e^{[\rho g_j(x)-1]} + e^{[\rho g_{j_1}(x)-1]} \right\} \\ &= \lim_{\rho \rightarrow +\infty} \frac{1}{\rho} \sum_{\substack{j=1 \\ j \neq j_1}}^p e^{[\rho g_j(x)-1]} + \lim_{\rho \rightarrow +\infty} \frac{1}{\rho} e^{[\rho g_{j_1}(x)-1]} \\ &= \lim_{\rho \rightarrow +\infty} \frac{1}{\rho} \sum_{\substack{j=1 \\ j \neq j_1}}^p e^{[\rho g_j(x)-1]} + \lim_{\rho \rightarrow +\infty} \frac{g_{j_1}(x) e^{\rho g_{j_1}(x)}}{e} \\ &= +\infty. \end{aligned}$$

According to Lemma 3.3, we can derive the conditions necessary for a point to become a feasible solution to a problem (MOOP) in the following theorem. $\square$

Lemma 3.4. *let $\rho_k > 0$, and $\lim_{k \rightarrow \infty} \rho_k = +\infty$, if $\bar{x} \in \overline{\lim_{k \rightarrow \infty} \widehat{X}_k}$, Then $\bar{x}$ is a feasible solution of (MOOP).*

Proof. Since $\bar{x} \in \overline{\lim_{k \rightarrow \infty} \widehat{X}_k}$, then there have a subsequence $\{k_n\}$ of $\{k\}$ that makes $\bar{x} \in \widehat{X}_{k_n}$, $n = \{1, 2, \dots\}$. Thus, by the definition of (MOOP) $_{w\rho}$ optimal solution it follows that

$$L(\bar{x}, \rho_k) \leq L(x, \rho_k), \quad \forall x \in X.$$

i.e.

$$\begin{aligned} &w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \dots + w_m f_m(\bar{x}) + \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(\bar{x})] \\ &\leq w_1 f_1(x) + w_2 f_2(x) + \dots + w_m f_m(x) + \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(x)], \quad \forall x \in X. \end{aligned} \quad (4)$$

Assuming that $\bar{x} \notin X$, by Lemma 3.3 we have $\lim_{n \rightarrow +\infty} \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(\bar{x})] = +\infty$.

For any $x \in X$, by Lemma 3.3 we have $\lim_{n \rightarrow +\infty} \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(x)] = 0$.

From the above two equations, for sufficiently large n ($n > n_0$), we have

$$\begin{aligned} &w_1 f_1(x) + w_2 f_2(x) + \dots + w_m f_m(x) + \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(x)] \\ &< w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \dots + w_m f_m(\bar{x}) + \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(\bar{x})]. \end{aligned}$$

This produces a contradiction with Eq. (4), Lemma 3.4 proof is complete. $\square$

3.3. Proof of convergence

In this section, we will prove the convergence of the weighted exponential penalty function method.

Theorem 3.1. $\lim_{k \rightarrow \infty} (\widehat{X}_k \setminus \widehat{X})$ is an empty set.

Proof. We use the inverse method. Assume that $\lim_{k \rightarrow \infty} (\widehat{X}_k \setminus \widehat{X}) \neq \emptyset$, and let $\bar{x} \in \lim_{k \rightarrow \infty} (\widehat{X}_k \setminus \widehat{X})$, then it exists a $k_0 > 0$, that makes for all $k > k_0$ there is $\bar{x} \in \widehat{X}_k \setminus \widehat{X}$.

(1) Assuming $\bar{x} \in X$, since $\bar{x} \notin \widehat{X}$, and combining Lemma 3.1-3.2, we have

$$\begin{aligned} & w_1 f_1(x) + w_2 f_2(x) + \cdots + w_m f_m(x) \\ & < w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \cdots + w_m f_m(\bar{x}), \text{ for some } x \in X. \end{aligned} \quad (5)$$

Moreover, since $\bar{x} \in \widehat{X}_k$, then we have

$$\begin{aligned} & w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \cdots + w_m f_m(\bar{x}) + \frac{1}{\rho_k} \sum_{j=1}^p \phi[\rho_k g_j(\bar{x})] \\ & \leq w_1 f_1(x) + w_2 f_2(x) + \cdots + w_m f_m(x) + \frac{1}{\rho_k} \sum_{j=1}^p \phi[\rho_k g_j(x)], \text{ for some } x \in X. \end{aligned}$$

When $k \rightarrow \infty$, using Lemma 3.3, the above inequality becomes

$$w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \cdots + w_m f_m(\bar{x}) \leq w_1 f_1(x) + w_2 f_2(x) + \cdots + w_m f_m(x).$$

This contradicts (5).

(2) Assuming that $\bar{x} \notin X$, by Lemma 3.3 we have $\lim_{k \rightarrow +\infty} \frac{1}{\rho_k} \sum_{j=1}^p \phi[\rho_k g_j(\bar{x})] = +\infty$.

Let $x' \in X$, from Lemma 3.3 we have $\lim_{k \rightarrow +\infty} \frac{1}{\rho_k} \sum_{j=1}^p \phi[\rho_k g_j(x')] = 0$, and hence for sufficiently large k we have

$$\begin{aligned} & w_1 f_1(x') + w_2 f_2(x') + \cdots + w_m f_m(x') + \frac{1}{\rho_k} \sum_{j=1}^p \phi[\rho_k g_j(x')] \\ & < w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \cdots + w_m f_m(\bar{x}) + \frac{1}{\rho_k} \sum_{j=1}^p \phi[\rho_k g_j(\bar{x})]. \end{aligned} \quad (6)$$

And because $\bar{x} \in \widehat{X}_k$, we have that

$$\begin{aligned} & w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \cdots + w_m f_m(\bar{x}) + \frac{1}{\rho_k} \sum_{j=1}^p \phi[\rho_k g_j(\bar{x})] \\ & \leq w_1 f_1(x') + w_2 f_2(x') + \cdots + w_m f_m(x') + \frac{1}{\rho_k} \sum_{j=1}^p \phi[\rho_k g_j(x')], \forall x' \in X. \end{aligned}$$

The above inequality contradicts (6). So, the theorem is proved. $\square$

Theorem 3.2. $\overline{\lim}_{k \rightarrow \infty} (\widehat{X}_k \setminus \widehat{X})$ is an empty set.

Proof. Use the inverse method. Assuming that $\overline{\lim}_{k \rightarrow \infty} (\widehat{X}_k \setminus \widehat{X}) \neq \emptyset$, it exists a subsequence $\{k_n\}$ of $\{k\}$ that makes $\bar{x} \in \widehat{X}_k \setminus \widehat{X}$.

- (1) Assuming that the variable $\bar{x} \in X$, since $\bar{x} \notin \hat{X}$, and combining Lemma 3.1 and Lemma 3.2, we have that

$$w_1 f_1(x) + w_2 f_2(x) + \cdots + w_m f_m(x) < w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \cdots + w_m f_m(\bar{x}), \text{ for some } x \in X.$$

Also, since $x \in X$, $\bar{x} \in X$, we have by Lemma 3.3(1) that

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(x)] = 0, \quad \lim_{n \rightarrow \infty} \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(\bar{x})] = 0.$$

So, the following inequality holds when n is sufficiently large

$$w_1 f_1(x) + w_2 f_2(x) + \cdots + w_m f_m(x) + \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(x)] < w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \cdots + w_m f_m(\bar{x}) + \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(\bar{x})], \quad (7)$$

and because $\bar{x} \in \widehat{X}_k$, we have that

$$w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \cdots + w_m f_m(\bar{x}) + \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(\bar{x})] \leq w_1 f_1(x) + w_2 f_2(x) + \cdots + w_m f_m(x) + \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(x)], \quad \forall x \in X. \quad (8)$$

Inequality (8) above contradicts (7). The theorem is proved.

- (2) Assuming $\bar{x} \notin X$, by Lemma 3 (2) we have $\lim_{n \rightarrow +\infty} \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(\bar{x})] = +\infty$.

By Lemma 3(1), for $\forall x \in X$, we have $\lim_{n \rightarrow +\infty} \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(x)] = 0$, and so we have the following inequality holds when there is a sufficiently large k_n , the

$$w_1 f_1(x) + w_2 f_2(x) + \cdots + w_m f_m(x') + \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(x)] < w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \cdots + w_m f_m(\bar{x}) + \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(\bar{x})], \quad (9)$$

and since $\bar{x} \in \widehat{X}_k$, the same inequality (8) holds, so inequality (9) contradicts inequality (8). Thus Theorem 3.2 is proved. $\square$

Theorem 3.3. $\lim_{k \rightarrow \infty} (\widehat{X}_k \setminus \hat{X})$ is an empty set.

The proof is derived from Theorem 3.1 and Theorem 3.2. $\square$

The following theorem shows that if the subsequence of the optimal solution of the problem $(MOOP)_{w\rho}$ converges to the feasible solution of the original problem $(MOOP)$, then the limit point of the penalty function at each point of the subsequence is zero.

Theorem 3.4. Let $\bar{x}_k$ be the minimum point of problem (3), $\{\bar{x}_{k_n}\}$ be a convergent subsequence of $\{\bar{x}_k\}$ and $\lim_{n \rightarrow \infty} \bar{x}_{k_n} \in X$, then there are

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi[\rho_{k_n} g_j(\bar{x}_{k_n})] = 0.$$

Proof. The same counterfactual approach is used. We assume that

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi [\rho_{k_n} g_j(\bar{x}_{k_n})] \neq 0.$$

Then there exists a subsequence $\{\bar{x}_{k_{n_r}}\}$, $r = 1, 2, \dots$ of sequence $\{\bar{x}_{k_n}\}$ such that

$$\lim_{r \rightarrow \infty} \frac{1}{\rho_{k_{n_r}}} \sum_{j=1}^p \phi [\rho_{k_{n_r}} g_j(\bar{x}_{k_{n_r}})] > \varepsilon, \text{ where } \varepsilon > 0,$$

it can also be deduced that the following limiting equation holds

$$\lim_{r \rightarrow \infty} \frac{1}{\rho_{k_{n_r}}} \sum_{j=1}^p \phi [\rho_{k_{n_r}} g_j(\bar{x}_{k_{n_r}})] > \varepsilon, \text{ where } \varepsilon > 0. \quad (10)$$

Since $\bar{x}_{k_{n_r}}$ is the minimum point of the problem $(MOOP)_{w\rho}$, there is

$$\begin{aligned} & w_1 f_1(\bar{x}_{k_{n_r}}) + w_2 f_2(\bar{x}_{k_{n_r}}) + \dots + w_m f_m(\bar{x}_{k_{n_r}}) + \frac{1}{\rho_{k_{n_r}}} \sum_{j=1}^p \phi [\rho_{k_{n_r}} g_j(\bar{x}_{k_{n_r}})] \\ & \leq w_1 f_1(x) + w_2 f_2(x) + \dots + w_m f_m(x) + \frac{1}{\rho_{k_{n_r}}} \sum_{j=1}^p \phi [\rho_{k_{n_r}} g_j(x)], \forall x \in X, \end{aligned}$$

since $\{\bar{x}_{k_n}\}$ is a convergent subsequence of $\{\bar{x}_k\}$ and $\lim_{n \rightarrow \infty} \bar{x}_{k_n} = \bar{x} \in X$, the above inequality can also be changed into the following form

$$\begin{aligned} & w_1 f_1(\bar{x}_{k_{n_r}}) + w_2 f_2(\bar{x}_{k_{n_r}}) + \dots + w_m f_m(\bar{x}_{k_{n_r}}) + \frac{1}{\rho_{k_{n_r}}} \sum_{j=1}^p \phi [\rho_{k_{n_r}} g_j(\bar{x}_{k_{n_r}})] \\ & \leq w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \dots + w_m f_m(\bar{x}) + \frac{1}{\rho_{k_{n_r}}} \sum_{j=1}^p \phi [\rho_{k_{n_r}} g_j(\bar{x})], \end{aligned}$$

taking the lower limit of $r \rightarrow \infty$ for the above inequality and combining it with inequality (10) we get

$$\begin{aligned} & \lim_{r \rightarrow \infty} \left[w_1 f_1(\bar{x}_{k_{n_r}}) + w_2 f_2(\bar{x}_{k_{n_r}}) + \dots + w_m f_m(\bar{x}_{k_{n_r}}) \right] + \varepsilon \\ & \leq w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \dots + w_m f_m(\bar{x}) + \lim_{r \rightarrow \infty} \frac{1}{\rho_{k_{n_r}}} \sum_{j=1}^p \phi [\rho_{k_{n_r}} g_j(\bar{x})], \end{aligned}$$

by the nature of limits, we have

$$\begin{aligned} & w_1 \lim_{r \rightarrow \infty} f_1(\bar{x}_{k_{n_r}}) + w_2 \lim_{r \rightarrow \infty} f_2(\bar{x}_{k_{n_r}}) + \dots + w_m \lim_{r \rightarrow \infty} f_m(\bar{x}_{k_{n_r}}) + \varepsilon \\ & \leq w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \dots + w_m f_m(\bar{x}) + \lim_{r \rightarrow \infty} \frac{1}{\rho_{k_{n_r}}} \sum_{j=1}^p \phi [\rho_{k_{n_r}} g_j(\bar{x})], \end{aligned}$$

and because $\lim_{n \rightarrow \infty} \bar{x}_{k_n} = \bar{x}$ it can be obtained that

$$\begin{aligned} & w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \dots + w_m f_m(\bar{x}) + \varepsilon \\ & \leq w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \dots + w_m f_m(\bar{x}) + \lim_{r \rightarrow \infty} \frac{1}{\rho_{k_{n_r}}} \sum_{j=1}^p \phi [\rho_{k_{n_r}} g_j(\bar{x})], \end{aligned}$$

so there's

$$\varepsilon \leq \lim_{r \rightarrow \infty} \frac{1}{\rho_{k_{n_r}}} \sum_{j=1}^p \phi [\rho_{k_{n_r}} g_j(\bar{x})], \quad (11)$$

also since $\bar{x} \in X$, by Lemma 3.3 (1) we have

$$\lim_{r \rightarrow \infty} \frac{1}{\rho_{k_{n_r}}} \sum_{j=1}^p \phi [\rho_{k_{n_r}} g_j(\bar{x})] = 0. \quad (12)$$

From Eq. (11) and Eq. (12) we get $\varepsilon \leq 0$ which contradicts $\varepsilon > 0$.

The theorem is proved. $\square$

Theorem 3.5. $\lim_{k \rightarrow \infty} \widehat{X_k} \subseteq \overline{\lim_{k \rightarrow \infty} \widehat{X_k}} \subseteq X$.

Proof. This is proved by Lemma 3.4. $\square$

Next, we use the following theorem to prove that the convergent subseries of the converted sequence of optimal solutions to the unconstrained single-objective optimization problem $(MOOP)_{w\rho}$ approximate the weakly efficient solution of the first problem $(MOOP)$.

Theorem 3.6. Let $\bar{x}_k$ be an optimal solution of $(MOOP)_{w\rho}$, if $\{\bar{x}_{k_n}\}$ is a convergent subsequence of $\{\bar{x}_k\}$ and $\lim_{n \rightarrow \infty} \bar{x}_{k_n} \in X$, then the limit point of $\bar{x}_{k_n}$, i.e., $\lim_{n \rightarrow \infty} \bar{x}_{k_n} = \bar{x}$, is a weakly efficient solution of $(MOOP)$.

Proof. let $\lim_{n \rightarrow \infty} \bar{x}_{k_n} = \bar{x}$, since $\bar{x}_{k_n} \in \widehat{X_k}$ for all $x_0 \in X$ such that the following inequality holds.

$$\begin{aligned} & w_1 f_1(\bar{x}_{k_n}) + w_2 f_2(\bar{x}_{k_n}) + \cdots + w_m f_m(\bar{x}_{k_n}) + \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi [\rho_{k_n} g_j(\bar{x}_{k_n})] \\ & \leq w_1 f_1(x_0) + w_2 f_2(x_0) + \cdots + w_m f_m(x_0) + \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi [\rho_{k_n} g_j(x_0)]. \end{aligned} \quad (13)$$

By Theorem 3.4 we have

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi [\rho_{k_n} g_j(\bar{x}_{k_n})] = 0. \quad (14)$$

Also, since x_0 is a feasible solution to $(MOOP)$, by Lemma 3.3 (1) we have

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_{k_n}} \sum_{j=1}^p \phi [\rho_{k_n} g_j(x_0)] = 0. \quad (15)$$

Combining the three expressions (13) (14) (15) we get

$$\begin{aligned} & w_1 \lim_{n \rightarrow \infty} f_1(\bar{x}_{k_n}) + w_2 \lim_{n \rightarrow \infty} f_2(\bar{x}_{k_n}) + \cdots + w_m \lim_{n \rightarrow \infty} f_m(\bar{x}_{k_n}) \\ & \leq w_1 f_1(x_0) + w_2 f_2(x_0) + \cdots + w_m f_m(x_0). \end{aligned}$$

And because $\lim_{n \rightarrow \infty} \bar{x}_{k_n} = \bar{x}$, the above equation of no becomes

$$w_1 f_1(\bar{x}) + w_2 f_2(\bar{x}) + \dots + w_m f_m(\bar{x}) \leq w_1 f_1(x_0) + w_2 f_2(x_0) + \dots + w_m f_m(x_0).$$

Therefore $\bar{x}$ is the optimal solution of the problem $(MOOP)_w$, and from Lemma 3.1-3.2, $\bar{x}$ is the weakly efficient solution of the problem $(MOOP)$. $\square$

In this way, the solution of problem $(MOOP)$ is converted to the solution of problem $(MOOP)_{w\rho}$. In the following, we can obtain a weakly effective solution to the original problem $(MOOP)$ by solving problem $(MOOP)_{w\rho}$.

4. Newton-GS Method

For the problem $(MOOP)_{w\rho}$, which is mainly a problem about unconstrained optimization, combined with the advantages of Newton method we use it in the next solution, the iterative formula of Newton's method can be expressed as follows (Ghazali *et al.* 2023):

$$x_{k+1} = x_k - \alpha_k d_k, \quad (16)$$

$$d_k = G_k^{-1} g_k. \quad (17)$$

Where d_k is the search direction, the step factor $\alpha_k = 1$, $G_k = \nabla^2 L(x_k)$, $g_k = \nabla L(x_k)$ and G_k^{-1} is the inverse matrix. Furthermore, we are aware that if G_k is a positive definite matrix, then it must have an inverse matrix, and the inverse matrix is also positive definite. Thus, the Newton direction will be a descent direction because it satisfies the following condition

$$g_k^T d_k = -g_k^T G_k^{-1} g_k < 0.$$

Because each step of Newton iterative method requires solving for the inverse matrix of G_k , which involves complex matrix operations, we transform Eq. (17) into the following linear system (Ghazali *et al.* 2020b)

$$G_k d_k = g_k. \quad (18)$$

In order to find an approximate value of d_k , the linear system (18) can be solved iteratively. Actually, there are several iterative methods that can be considered (Lung & Sulaiman 2020; Sulaiman *et al.* 2007). However, this paper attempt to solve the linear system by using the Newton-Gauss-Seidel (Newton-GS) iterative method.

Thus, we obtain the steps of the algorithm for Newton's method and weighted EPF method for the problem $(MOOP)_{w\rho}$ as follows:

Table 1. Algorithm: Newton-GS method and weighted exponential penalty function method

I	For $w_1 = 0: \sigma: 1$
	For $w_2 = 0: \sigma: 1 - w_1$
	...
	For $w_{m-1} = 0: \sigma: 1 - \sum_{i=1}^{m-2} w_i$
II	Calculate the expression of the objective function
	$L(x, \rho) = w_1 f_1(x) + w_2 f_2(x) + \dots + (1 - \sum_{i=1}^{m-1} w_i) f_m(x) + \frac{1}{\rho} \sum_{j=1}^p \phi[\rho g_j(x)]$
III	Given initial values x_0, ρ_0 and an accuracy threshold $\varepsilon_1 = 10^{-5}$, $\varepsilon_2 = 10^{-10}$ and let $k = 0$.
	Compute the gradient g_k and the Hessian matrix G_k .

Table 1 (Continued)

IV	If $\ g_k\ < \varepsilon_1$, i.e., at this point the gradient value is close to 0, the extreme point is reached and the calculation stops. Otherwise go to a.
	a. Calculate the matrix D, E, U , Satisfies $G_k = D - E - U$.
	b. Compute the search direction $d_{k+1} = (D - E)^{-1} U d_k + (D - E)^{-1} g$.
	c. Calculate the convergence criteria, that is, if $\ d_{k+1} - d_k\ < \varepsilon_2$, then go to V, otherwise go to b.
V	Getting a new calculation point: $x_{k+1} = x_k + \alpha_k d_k$.
VI	Let $k := k + 1$ and go to III.
	End
	...
	End
	End

5. Numerical Experiments

Below we give some example calculations to demonstrate the effectiveness of the algorithm proposed in this paper.

Example 5.1 (Jain & Deb 2014):

$$\begin{aligned}
 \min f_1(x) &= (1 + g(x)) \cos\left(\frac{x_1\pi}{2}\right) \cdots \cos\left(\frac{x_{M-1}\pi}{2}\right), \\
 \min f_2(x) &= (1 + g(x)) \cos\left(\frac{x_1\pi}{2}\right) \cdots \sin\left(\frac{x_{M-1}\pi}{2}\right), \\
 &\vdots \\
 \min f_M(x) &= (1 + g(x)) \sin\left(\frac{x_1\pi}{2}\right), \\
 g(x) &= \sum_{i=M}^n (x_i - 0.5)^2, \\
 s. t. \quad c(x) &= \sum_{i=1}^M (f_i(x) - \delta)^2 - r^2 \geq 0, \\
 x &\in [0,1],
 \end{aligned}$$

where $M = 2$, $\delta = \frac{1}{M} \sum_{i=1}^M f_i(x)$, $r = 0.225$. Firstly, the Example 5.1 is converted to a single-objective optimization problem by the weighting method as follows, defined as $(MOOP1)_w$.

$$\begin{aligned}
 \min F(x) &= w_1 f_1(x) + w_2 f_2(x), \\
 s. t. \quad c(x) &= \sum_{i=1}^M (f_i(x) - \delta)^2 - r^2 \geq 0, \\
 x &\in [0,1],
 \end{aligned}$$

where, w is the weight coefficient, and $\sum_{i=1}^2 w_i = 1$.

The EPF method is then used to converted $(MOOP1)_w$ into an unconstrained optimization problem as follows, defined as $(MOOP1)_{w\rho}$.

$$\min L(x, \rho) = w_1 f_1(x) + w_2 f_2(x) + \frac{1}{\rho} \sum_{j=1}^p \phi[\rho c(x)].$$

where, $\rho > 0$ is the penalty parameter and $\sum_{j=1}^p \phi[\rho c(x)] = \sum_{t=1}^n e^{\rho(-c(x))^{-1}}$, is the exponential function.

Choose the initial point as $x_0 = (0.5, 0.5, \dots, 0.5)^T$, choose the weight coefficients w to be changed once in a cycle $\sigma = 0.01$, and use Newton algorithm to get a set of optimal solutions of $(MOOP1)_{w\rho}$ under different weight coefficients using the MATLAB software, and according to Theorem 3.6, we can know that this set of optimal solutions is the $(MOOP)$

problem of the pareto weakly effective solution, Figure 1 show the pareto front when the number of decision variables $n = 10$ and $n = 20$.

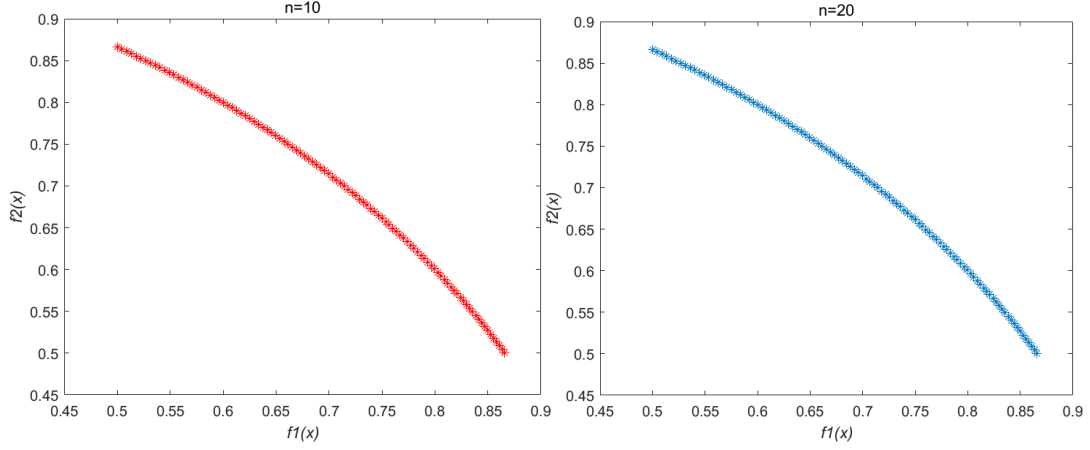


Figure 1. Solution results for Example 5.1 at $n = 10$ and $n = 20$.

Example 5.2 (Zou *et al.* 2008)

$$\begin{aligned} \min f_1(x) &= (x_1 - 1)^2 + \sum_{t=2}^n x_t^2, \\ \min f_2(x) &= x_1^2 + (x_2 - 1)^2 + \sum_{t=3}^n x_t^2, \\ \min f_3(x) &= x_1^2 + x_2^2 + (x_3 - 1)^2 + \sum_{t=4}^n x_t^2, \\ \text{s.t. } x_t - 1000 &\leq 0, \quad t = 1, \dots, n, \\ -x_t - 1000 &\leq 0, \quad t = 1, \dots, n. \end{aligned}$$

For Example 5.2, we use the same method as Example 5.1 to transform it. We choose the initial point as $x_0 = (1, 1, \dots, 1)^T$, in order to keep the number of effective solutions consistent with Example 5.1, we choose the weight coefficients w to be changed once in a cycle with the amount $\sigma = 0.05$, and use Newton algorithm to get a set of optimal solutions of $(MOOP2)_{w\rho}$ under different weight coefficients using MATLAB software, again according to Theorem 3.6 it can be seen that this set of optimal solutions is the pareto weakly efficient solution of the $(MOOP2)$ problem, Figure 2 and Figure 3 show the Pareto front in each direction when the number of decision variables $n = 10$ and $n = 20$.

6. Conclusions

This paper focuses on the solution method of multi-objective constrained optimization problems. We combine the linear weighted method and the EPF method to propose a new linear weighted EPF method. The method is used to transform a multi-objective problem with constraints into an unconstrained single-objective optimization problem. Meanwhile, we prove the convergence of the method. Based on the converted unconstrained optimization problem, we mainly use Newton's method to solve it, in which the weights of the weighting method are continuously changed by means of a loop to obtain a set of Pareto weakly effective solutions. The solutions of the above examples 5.1-5.2 also illustrate the effectiveness of our proposed method. Overall, this article combines the weighting method, the exponential penalty function method and Newton method to obtain a new method for solving multi-objective problems. Numerical experiments show the usefulness and advantages of the algorithm in increasing the solution quality, and how to innovate the method to make the computation more efficient, or to

compute the ultra-large-scale multi-objective optimization problems will be the direction of our next further research.

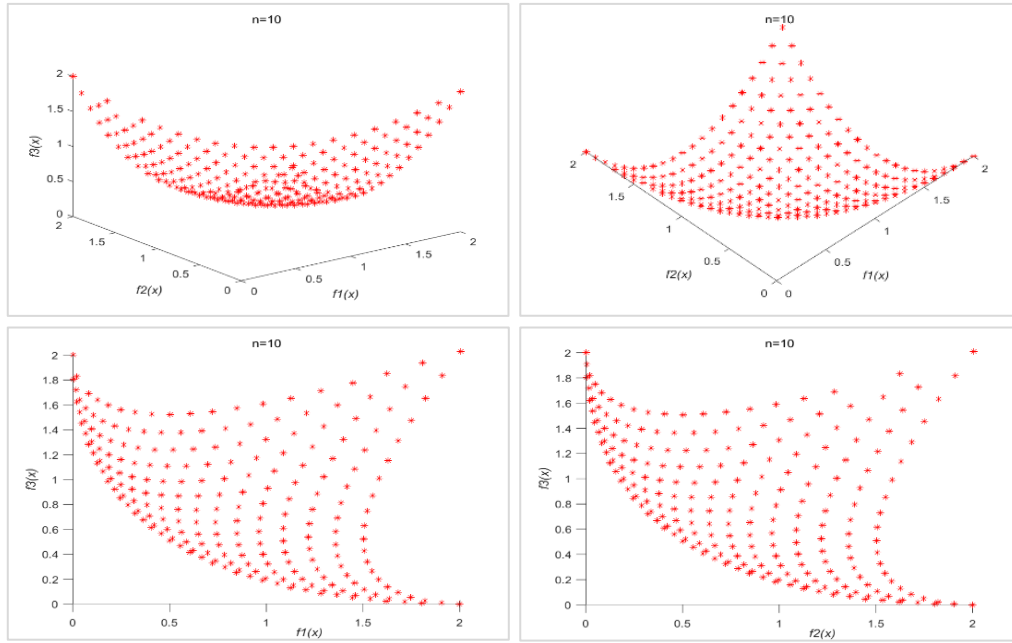


Figure 2. Solution results for Example 5.2 at $n = 10$.

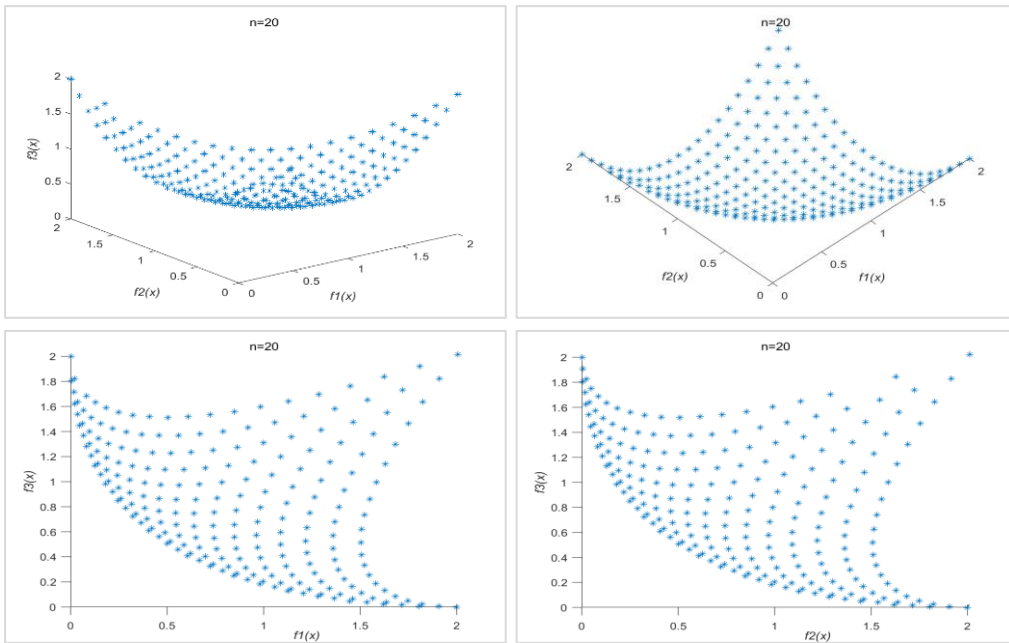


Figure 3. Solution results for Example 5.2 at $n = 20$.

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