

DYNAMICS OF DEGENERATE QUADRATIC LOTKA-VOLTERRA MAPPINGS ACTING IN A FOUR-DIMENSIONAL SIMPLEX AND BIGRAPHS

ULUGBEK RAKHIMJONOVICH MUMINOV, DILFUZA BAKHROMOVNA ESHMAMATOVA,
LOLA DJAVLANOVNA SHARIPOVA* & YORKIN ABDUKHAKIMOVICH ISLAMOV

ABSTRACT

The work is devoted to the problem of studying the trajectories of points under the action of the Lotka-Volterra operator acting in a four-dimensional simplex. We show the connection between oriented bigraphs and Lotka-Volterra mappings, and with the skew-symmetric matrices corresponding to them. The connection between such type operators and the corresponding skew-symmetric matrices is considered for the first time. In the paper, we introduce a definition of bigraphs as oriented graphs whose vertices are separated into two groups. Based on this, bigraphs, the vertices of which are separated as follows: one vertex is assigned to one group, and the remaining four are assigned to another group, are divided into three subclasses, taking into account the isomorphy of the bigraphs included in these groups. One representative from each isomorphic group was examined for the presence of fixed points of operators corresponding to them, and the characters of these points are established.

Keywords: attractor; eigenvalue; fixed point; neutral fixed point; repeller; saddle point

1. Introduction

An important scientific problem in natural science is to solve the problem of predicting the behavior of an object under study over time or space, knowing its initial state. This problem boils down to finding a specific law that allows us to determine the future state of the object based on its initial state.

Of course, predicting the evolution of an object is a mathematical problem. In our case, we will not be analyzing systems and objects in general, but rather so-called discrete dynamical Lotka-Volterra systems.

It is known that in the problems of physics and mechanics, continuous dynamical systems are the subject of the mathematical model of the process under study. Mainly, autonomous differential equations and partial differential equations should be taken into account (Hubbard & West 1995; Chau 2019; Betounes 2010).

It should also be noted that this is widely used by biologists, epidemiologists, and environmentalists (Srtrogatz 2015; Martcheva 2015; Brauer & Castillo-Chávez 2001). Additionally, it is used for the tasks of optimizing transport (Amraoui *et al.* 2023; Yang *et al.* 2021). But it turned out that for many situations in these areas, it is preferable to consider changes over time, from year to year and from generation to generation. In transport problems, this time period is every month (Amraoui *et al.* 2023; Yang *et al.* 2021). And of course, as we noted above in the description and study of such problems, as an evolutionary operator, it is advisable to use a function that expresses the state of the system at a certain point in time through its state at a previous moment.

In Ganikhodzhaev (1993) a discrete quadratic stochastic operator that preserves a finite-dimensional simplex was introduced. Works by Ganikhodzhaev (1994), Eshmamatova *et al.* (2023b), and Eshmamatova and Ganikhodzhaev (2024) are devoted to the study of quadratic stochastic operators using elements of tournament theory, in particular, complete directed graphs, and the construction of the Lyapunov function. Non-degenerate Lotka-Volterra mappings are also studied using tournaments and constructing the Lyapunov function. In Eshmamatova

(2023) and Eshmamatova and Yusupov (2024), a concept of a degenerate Lotka-Volterra mapping was introduced, and several of these were considered that preserve two-dimensional and three-dimensional simplices. In Eshmamatova (2023), Eshmamatova (2024b), Eshmamatova *et al.* (2023a), and Ganikhodzhaev *et al.* (2024), degenerate Lotka-Volterra mappings acting in S^2 and S^3 are proposed as discrete compartment models describing the dynamics of airborne viral diseases. Unlike these previous works, we present for the first time discrete dynamic Lotka-Volterra mappings with a degenerate skew-symmetric matrix that preserves a four-dimensional simplex. We have identified among them those belonging to the bigraph class $B_{1,4}$. Here, in contrast to the previous works, we give a new method for constructing routes using a signature. After considering all the signatures, we divide the simplex, plot the routes of the trajectories of the interior points and investigate their dynamics. We also find fixed points and explore their characters.

2. Preliminary Definitions

Let us start with the description of our object Ganikhodzhaev (1993) and Tadzhieva *et al.* (2024). It is known Ganikhodzhaev (1993)-Tadzhieva *et al.* (2024), Ganikhodzhaev *et al.* (2024) that in the space $\mathbb{R}^m$, the state of a biological system consisting of m species can be determined by the vector $x = (x_1, x_2, \dots, x_m)$, where $x_i \geq 0$, $\sum_{i=1}^m x_i = 1$. Here x_i represents the fraction of the i -th species. The vector $x = (x_1, x_2, \dots, x_m)$ itself is the probability distribution of species characters. More specifically, the set of these vectors can be represented as an $(m - 1)$ -dimensional simplex:

$$S^{m-1} = \left\{ x \in \mathbb{R}^m : x = (x_1, x_2, \dots, x_m) : x_i \geq 0, \sum_{i=1}^m x_i = 1 \right\}. \quad (1)$$

It is clear that a simplex is always a convex compact set Ganikhodzhaev (1993). The vertices of the simplex are $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, ..., $e_m = (0, 0, 0, \dots, 1)$ (Ganikhodzhaev 1993, 1994; Tadzhieva *et al.* 2024). Denote by riS^{m-1} the relative interior of the simplex S^{m-1} , that is, the set of relative interior points of the simplex.

Let $I = \{1, 2, \dots, m\}$, and its nonempty subset $\alpha \subset I$.

Definition 2.1. (Ganikhodzhaev *et al.* 2024) The convex hull of the vertices e_i , $i \in \alpha$ of the simplex S^{m-1} is called a face of the simplex and is denoted by Γ_α , i.e. $\Gamma_\alpha = co\{e_i\}_{i \in \alpha}$.

If $|\alpha|$ denotes the number of elements of the set α , then the face Γ_α can be represented as $\Gamma_\alpha = co\{e_i\}_{i \in \alpha}$, and its dimension will be equal to $dim \Gamma_\alpha = |\alpha| - 1$.

In Ganikhodzhaev (1993), the following quadratic stochastic operator was introduced, which preserves S^{m-1} :

$$(Vx)_k = \sum_{i,j=1}^m P_{ij,k} x_i x_j, \quad k \in \{1, 2, \dots, m\}, \quad (2)$$

where the coefficients of this operator satisfy the conditions

$$P_{ij,k} = P_{ji,k} > 0, \quad \sum_{k=1}^m P_{ij,k} = 1, \quad i, j \in \{1, 2, \dots, m\} \quad (3)$$

According to Ganikhodzhaev (1993) and Ganikhodzhaev (1994), we define the Lotka-Volterra mapping.

Definition 2.2. (Ganikhodzhaev 1994) If the coefficients of $V : S^{m-1} \rightarrow S^{m-1}$ described by equations (2) satisfy the conditions $P_{ij,k} = 0$ when $k \notin \{i, j\}$, then V is called the Lotka-Volterra mapping.

According to Definition 2.2, Ganikhodzhaev (1993) and Ganikhodzhaev (1994), the Lotka-Volterra mapping can be represented as

$$V : x'_k = x_k \left(1 + \sum_{i=1}^m a_{ki} x_i \right), \quad k = 1, \dots, m. \quad (4)$$

This mapping is completely determined by the skew-symmetric matrix $A = (a_{ij})$, whose elements satisfy the conditions $a_{ij} = -a_{ji}$ and $|a_{ki}| \leq 1$ for all $i, j \in \{1, 2, \dots, m\}$. We know, that the Lotka-Volterra operator is a homeomorphism, that is, there exists a continuous inverse operator V^{-1} for this operator (Ganikhodzhaev 1993, 1994; Tadzhiyeva *et al.* 2024).

Let $x^0 \in S^{m-1}$. Then the sequence obtained as follows

$$x^0, x^{(1)} = V(x^0), x^{(2)} = V(Vx^0) = V^2(x^0), \dots, x^{(n)} = V(V \dots (Vx^0)) = V^n(x^0), \dots$$

is called the trajectory or orbit of this point. For an arbitrary point of the simplex $x^0 \in S^{m-1}$, we define the set of limit points of its positive trajectory as $\omega(x^0) = \{x^0, x^{(1)}, \dots\}'$, and the set of limit points of its negative trajectory as $\alpha(x^0) = \{x^0, x^{(-1)}, x^{(-2)}, \dots\}'$.

Now, for operator (4), we introduce the concept of a fixed point, as described by Devaney (2020) and Devaney (2003).

Definition 2.3. (Devaney 2020, 2003) Let $x \in S^{m-1}$. If the condition $Vx = x$ is satisfied, then the point x is called fixed.

All the results below are based on the following theorem from Ganikhodzhaev (1993), Ganikhodzhaev (1994), and Tadzhiyeva *et al.* (2024). Therefore, we have decided to provide it in full, for the sake of completeness.

Theorem 2.4. (Ganikhodzhaev 1993, 1994; Tadzhiyeva *et al.* 2024) Let $A = (a_{ki})$ be a skew-symmetric matrix. Then, the sets

$$P = \{x \in S^{m-1} : Ax \geq 0\} \neq \emptyset, \quad Q = \{x \in S^{m-1} : Ax \leq 0\} \neq \emptyset$$

are convex sets, that consist of fixed points.

In Tadzhiyeva *et al.* (2024) it was proved that the set P consists of a set of limit points of a negative trajectory, i.e. $\alpha(x^0) \subset P$, and the set Q consists of a set of limit points of a positive trajectory, i.e. $\omega(x^0) \subset Q$. That was this is how the dynamics of the trajectories of internal points was studied. In this paper, we will divide the simplex into parts and find these sets based on their signatures. Using signatures and partitioning the simplex provides us with a more detailed understanding of the route of the interior points of the simplex.

For an operator $f_k(x_1, x_2, \dots, x_m)$, the Jacobi matrix has the following form (Gantmacher 1960):

$$J(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_m} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_m} \end{pmatrix}. \quad (5)$$

Let $x^* = (x_1, x_2, \dots, x_m) \in S^{m-1}$ be a fixed point of the system of the form (4). If

$$|J(x^*) - \lambda E| = 0, \quad (6)$$

then the numbers λ_i , $i \in \{1, 2, \dots, m\}$, which are solutions to the equation, are called the eigenvalues of the fixed point Devaney (2020). Here E is a unit matrix, λ is an eigenvalue.

Now let us recall the definitions from Devaney (2020), and Devaney (2003). With the help of these definitions, we can analyze the spectrum of the Jacobian, and characterize fixed points by analyzing the spectrum.

Definition 2.5. (Devaney 2020) If for a fixed point $x^* = (x_1, x_2, \dots, x_m) \in S^{m-1}$, all the eigenvalues λ_i , $i \in \{1, 2, \dots, m\}$ of the Jacobi matrix $J(x)$ are less than one in absolute value, then this fixed point is called attracting.

Definition 2.6. (Devaney 2020) If for a fixed point $x^* = (x_1, x_2, \dots, x_m) \in S^{m-1}$, all the eigenvalues λ_i , $i \in \{1, 2, \dots, m\}$ of $J(x)$ are greater than one, then this fixed point is called repulsive.

In all other cases, a fixed point is said to be a saddle point.

The dynamic properties of various types of dynamical systems, including linear and nonlinear ones, are described in works Devaney (2020), Devaney (2003), Ichiro (2001), Oprocha and Wilczynski (2001), Shapovalov (2000), and Holmgren (1996). Unlike these works, the present work is devoted to discrete Lotka-Volterra dynamical systems and their relationship with bigraphs. It is known Tadzhiyeva *et al.* (2024), that each skew-symmetric matrix A with the signs of elements a_{ki} can be matched with a directed graph. If all the elements of the matrix $a_{ki} \neq 0$ at $k \neq i$, then the directed graph is called a tournament. The dynamics of the trajectories of the Lotka-Volterra mapping in the case of tournaments is studied in Ganikhodzhaev (1993), Ganikhodzhaev (1994), and Tadzhiyeva *et al.* (2024).

Definition 2.7. A directed graph is called a bipartite graph or bigraph if its vertices can be divided into two nonempty classes I and II in such a way that the vertices of any edge belong to different classes. A bigraph is called complete if any vertex from the class I is adjacent to every vertex from the class II .

3. Trajectory dynamics under the influence of the Lotka-Volterra operator

In this paper, we consider degenerate cases of Lotka-Volterra mappings corresponding to bipartite graphs. Previously, in Eshmamatova (2024b), and Eshmamatova (2024a), non-degenerate Lotka-Volterra mappings corresponding to tournaments – oriented graphs – were studied, and degenerate mappings corresponding to partially oriented graphs were studied in Eshmamatova *et al.* (2023a), and Eshmamatova *et al.* (2024). In these works, studies of the asymptotic behavior of trajectories were carried out using the construction of the Lyapunov function, as well as the analysis of the Jacobian spectrum. In this paper, we investigate those cases that have not been previously considered. The vertices of the graphs corresponding to these mappings can be divided into two subgroups. In this paper, we consider a four-dimensional simplex, so we divide the vertices of the simplex as follows: one vertex is assigned to one group, and the remaining vertices to another subgroup. It turned out that there are 16 bigraphs of this type, corresponding to degenerate cases of Lotka-Volterra mappings operating in a four-dimensional simplex. We managed to classify and analyze them by splitting the simplex into parts taking into account the signature. Then we found for each signature the sets of limit points of positive and negative trajectories with the help of Jacobian spectrum analysis.

Here we will highlight isomorphic graphs. To do so, we need to recall the concept of graph isomorphism. The concept of tournaments isomorphism is given in Harary and Palmer (1973),

Chartrand *et al.* (2011), Koh *et al.* (2007), Griffin (2023), and Cioba and Murty (2022). Formally, the isomorphism of orgraphs is defined as for undirected graphs: two directed graphs G and G' are said to be isomorphic (or equal) if a one-to-one correspondence can be established between their elements of the same type that preserves the incidence relation. It should be noted that if two orgraphs are isomorphic, then the undirected graphs that are their symmetrizations will also be isomorphic. However, the converse is not always true.

It turned out that for each skew-symmetric matrix, there is a partially oriented bigraph that can be matched to it. Consider bigraphs, in which n vertices belong to the first class and the remaining vertices belong to the second class. We will denote such bigraphs by $B_{n, m-n}$. According to Definition 1.5, below we will consider the class of digraphs $B_{1,4}$ on the simplex S^4 , as shown in Figure 1. In Figure 1, there are bigraphs with one vertex belonging to the first class and the remaining four vertices belonging to another class. These bigraphs are classified into three groups. In total, there are 16 cases where the vertex $\{e_1\}$ belongs to the class I , and the remaining vertices $\{e_2, e_3, e_4, e_5\}$ belong to the class II . Among these 16 cases, we select eight isomorphic bigraphs from the first group, six from the second group, and two from the third group. The vertices $\{e_2, e_3, e_4, e_5\}$, belonging to the same group, do not have any directions between them.

4. Main results

Now let us consider one representative from each group of bigraphs and corresponding skew-symmetric matrices, and Lotka-Volterra operators that act in

$$S^4 = \left\{ x = (x_1, x_2, x_3, x_4, x_5) \in R^5 : \sum_{i=1}^5 x_i = 1, x_i \geq 0 \right\}.$$

Let e_1, e_2, e_3, e_4, e_5 be vertices of S^4 , $I = \{e_1\}$, $II = \{e_2, e_3, e_4, e_5\}$, and let the matrix A_1 have the form

$$A_1 = \begin{pmatrix} 0 & a & b & c & -d \\ -a & 0 & 0 & 0 & 0 \\ -b & 0 & 0 & 0 & 0 \\ -c & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 \end{pmatrix}, 0 < a, b, c, d \leq 1. \quad (7)$$

The corresponding bigraph G_A is shown in Figure 2.

In this case, the operator $V_1 : S^4 \rightarrow S^4$ defined by the matrix A_1 have the following form:

$$V_1 : \begin{cases} x_1' = x_1(1 + ax_2 + bx_3 + cx_4 - dx_5), \\ x_2' = x_2(1 - ax_1), \\ x_3' = x_3(1 - bx_1), \\ x_4' = x_4(1 - cx_1), \\ x_5' = x_5(1 + dx_1). \end{cases} \quad (8)$$

First, we find the fixed points of the operator V_1 Devaney (2020). To do this, we solve the system of equations $V_1 x = x$. The solutions to this system are the vertex $e_1 = (1, 0, 0, 0, 0) \in S^4$ and the face spanned by the vertices of the e_2, e_3, e_4 and e_5 , i.e.

$$\Gamma_{2345} = \left\{ (0, x_2, x_3, x_4, x_5) \in S^4 : 0 < x_2, x_3, x_4, x_5 \leq 1, \sum_{i=2}^5 x_i = 1 \right\}.$$

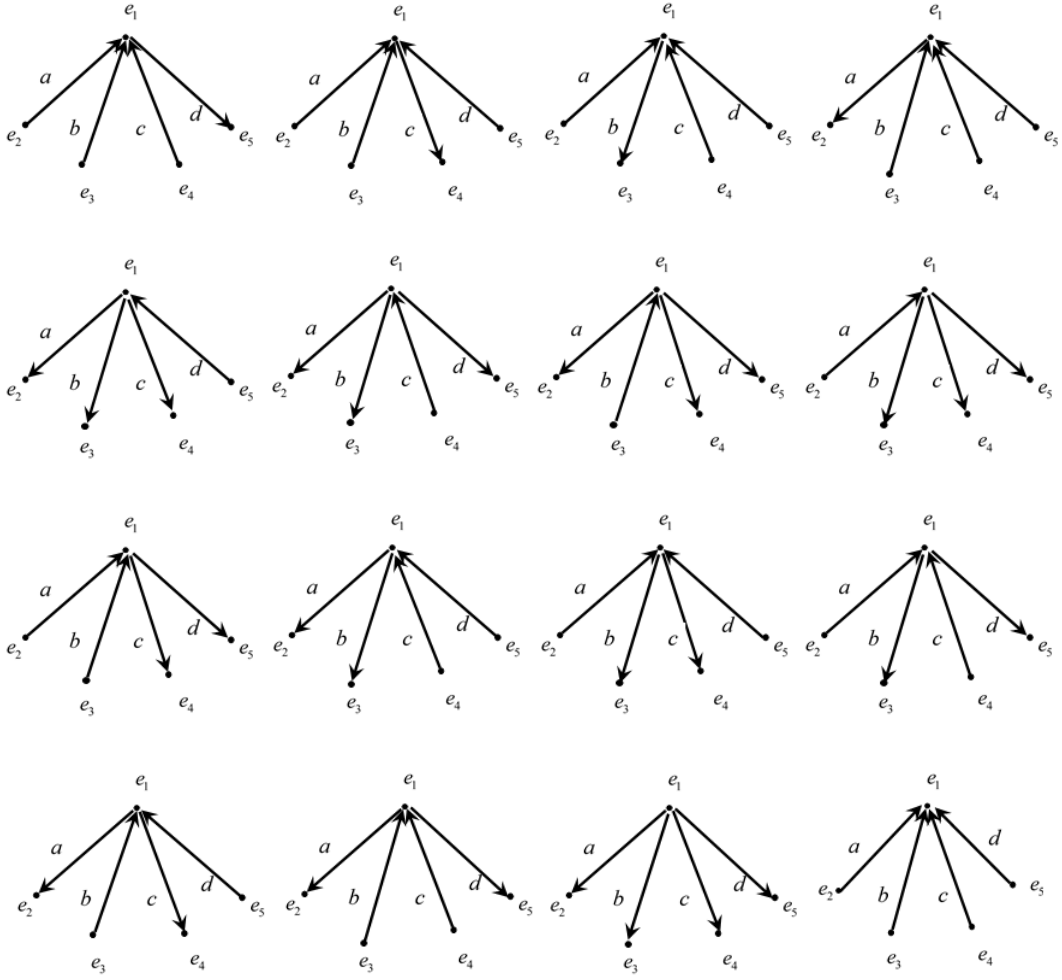


Figure 1: All bigraphs $B_{1,4}$.

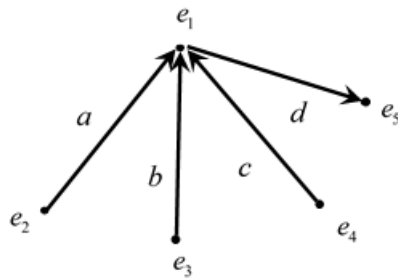


Figure 2: The partially oriented bigraph corresponding to the matrix A_1

The next theorem describes the characters of the fixed points.

Theorem 4.1. *The fixed point $e_1 = (1, 0, 0, 0, 0)$ of the simplex is the saddle point.*

Proof. We need to find the eigenvalues for the point $e_1 = (1, 0, 0, 0, 0)$ using the Jacobi matrix according to equality (6). In other words, we need to solve an equation of the form

$$(1 - \lambda)^3 (1 + a - \lambda) (1 - b - \lambda) = 0.$$

The solutions have the form: $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\lambda_4 = 1 + a$, $\lambda_5 = 1 - b$. Since $|\lambda_4| > 1$, $|\lambda_5| < 1$, we can conclude that one eigenvalue is greater than one in absolute value, and the second eigenvalue is modulo less than one. This means that the vertex e_1 is a saddle point. $\square$

We skip the eigenvalues that are equal in modulus to one, as described in Eshmamatova *et al.* (2023b), and Eshmamatova and Ganikhodzhaev (2024). Now consider the first equation from the right-hand side of system (5), which is $ax_2 + bx_3 + cx_4 - dx_5 = 0$. The solutions to this equation on the face Γ_{2345} form a triangular plane ΔABC . All points of this plane are neutral fixed points. Here $A \left(0, \frac{d}{a+d}, 0, 0, \frac{a}{a+d}\right)$, $B \left(0, 0, 0, \frac{d}{c+d}, \frac{c}{c+d}\right)$, $C \left(0, 0, \frac{d}{b+d}, 0, \frac{b}{b+d}\right)$.

We know Eshmamatova *et al.* (2023b), and Eshmamatova and Ganikhodzhaev (2024) that the trajectory begins in the set $P = \{x \in S^4 : Ax \geq 0\}$ and ends in the set $Q = \{x \in S^4 : Ax \leq 0\}$. Using matrix (7) and conditions of Theorem 2.4, we obtain the following systems of inequalities:

$$\left\{ \begin{array}{l} ax_2 + bx_3 + cx_4 - dx_5 \geq 0, \\ -ax_1 \geq 0, \\ -bx_1 \geq 0, \\ -cx_1 \geq 0, \\ dx_1 \geq 0, \end{array} \right. \quad \left\{ \begin{array}{l} ax_2 + bx_3 + cx_4 - dx_5 \leq 0, \\ -ax_1 \leq 0, \\ -bx_1 \leq 0, \\ -cx_1 \leq 0, \\ dx_1 \leq 0. \end{array} \right.$$

The solutions of these systems of inequalities consist of sets $P = co\{e_2, e_3, e_4, A, B, C\}$ and $Q = co\{e_5, A, B, C\}$, respectively. That is, the set P consists of a convex hull of vertices e_2, e_3, e_4 and fixed points A, B and C , the set Q consists of the convex hull of the vertex e_5 and the fixed points A, B and C (see Figure 3).

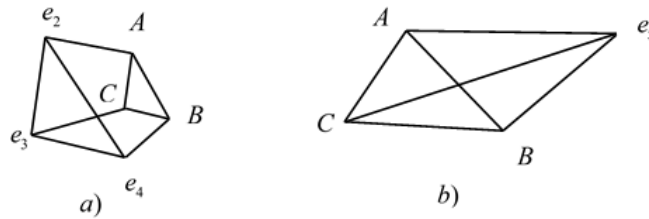


Figure 3: a) The set P , i.e. the starting points of the trajectories, b) the set Q , i. e. the end points of the trajectories.

Introduce the following notations:

$$\Omega_1 = co\{e_2, e_3, e_4, A, B, C\} / \Gamma_{ABC}, \quad \Omega_2 = co\{e_5, A, B, C\} / \Gamma_{ABC}.$$

Theorem 4.2. *All points of Ω_1 are repulsive, and all points of Ω_2 are attracting.*

Proof. Consider a set consisting of the face of Γ_{2345} , that is, the fixed points. In order to find the eigenvalues, we will create the equation:

$$|J(\Gamma_{2345}) - \lambda E| = (1 + ax_2 + bx_3 + cx_4 - dx_5 - \lambda)(1 - \lambda)^4 = 0.$$

Then the eigenvalues for the set of points Γ_{2345} will have the form: $\lambda_1 = 1 + ax_2 + bx_3 + cx_4 - dx_5$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 1$. We are interested in the sign of the polynomial $ax_2 + bx_3 + cx_4 - dx_5$. The sign of this polynomial is positive in the polyhedron Ω_1 , which means that $|\lambda_1| > 1$. By Definition 2.6, all points of this polyhedron are repulsive. Conversely, in the polyhedron Ω_2 , the sign of the polynomial is negative because $|\lambda_1| < 1$. By Definition 2.5, all the points of Ω_2 are attractors. $\square$

As we explained above in the second part of the paper, an arbitrary point x of the simplex S^{m-1} passes to another point of the simplex using the operator $Vx = (x'_1, x'_2, \dots, x'_m)$. When performing the inequality $x'_k > x_k$, $k \in \{1, 2, \dots, m\}$ we assume that the k -th coordinate of the point increases and instead of it we use the sign “+”. If we have the inequality $x'_k < x_k$, then the k -th coordinate of the point decreases instead of it we use the sign “-”. The set of characters obtained in this way is called a signature and is denoted by δ . For operator (8), we have only two signatures $\delta_1 = (- - - - +)$ and $\delta_2 = (+ - - - +)$. The systems of inequalities corresponding to these signatures are the following:

$$\left\{ \begin{array}{l} ax_2 + bx_3 + cx_4 - dx_5 \leq 0, \\ -ax_1 \leq 0, \\ -bx_1 \leq 0, \\ -cx_1 \leq 0, \\ dx_1 \geq 0, \end{array} \right. \quad \left\{ \begin{array}{l} ax_2 + bx_3 + cx_4 - dx_5 \geq 0, \\ -ax_1 \leq 0, \\ -bx_1 \leq 0, \\ -cx_1 \leq 0, \\ dx_1 \geq 0. \end{array} \right.$$

The solutions of the inequalities corresponding to these signatures have the form:

$$T_1 = \text{co}\{e_1, e_5, A, B, C\} \text{ and } T_2 = \text{co}\{e_1, e_2, e_3, e_4, A, B, C\}.$$

We are interested in the invariance of these sets, and in order to achieve this goal, we have obtained the following

Lemma 4.3. For sets T_1 and T_2 , the following holds: $V(T_1) \subset T_1$, $V^{-1}(T_2) \subset T_2$.

Proof. Let $x \in T_1$. In this case, the inequality $ax_2 + bx_3 + cx_4 - dx_5 \leq 0$ holds. This implies that

$$\begin{aligned} ax_2' + bx_3' + cx_4' - dx_5' &= ax_2(1 - ax_1) + bx_3(1 - bx_1) + cx_4(1 - cx_1) \\ &\quad - dx_5(1 + dx_1) \\ &= ax_2 + bx_3 + cx_4 - dx_5 - (a^2x_1x_2 + b^2x_1x_3 \\ &\quad + c^2x_1x_4 + d^2x_1x_5) \leq 0. \end{aligned}$$

Hence, $x' = Vx \in T_1$.

Now, for the inverse operator V^{-1} , we prove the invariance of the set T_2 . Let us suppose that $x \in T_2$. In this case, the inequality $ax_2' + bx_3' + cx_4' - dx_5' \geq 0$ holds. This inequality, along

with the following relation

$$\begin{aligned} ax_2' + bx_3' + cx_4' - dx_5' &= ax_2(1 - ax_1) + bx_3(1 - bx_1) + cx_4(1 - cx_1) \\ &\quad - dx_5(1 + dx_1) \\ &= ax_2 + bx_3 + cx_4 - dx_5 - (a^2x_1x_2 + b^2x_1x_3 \\ &\quad + c^2x_1x_4 + d^2x_1x_5) \geq 0 \end{aligned}$$

implies the inequality $ax_2 + bx_3 + cx_4 - dx_5 \geq 0$. Then we get $V^{-1}(T_2) \subset T_2$.

Therefore, T_1 is invariant for a positive trajectory, and T_2 is invariant for a negative trajectory. We know that the intersection of the polyhedra corresponding to the signatures δ_1 and δ_2

$$T_1 \cap T_2 = \text{co}\{e_1, A, B, C\}$$

consists of a polyhedron, and for all points $x \in T_1 \cap T_2$, the equality $x' = x$ holds. Then, the sequence $\{x_1^{(n)}\}$, $n \in \mathbb{Z}$ reaches its single maximum value only in the polyhedron $T_1 \cap T_2$.

It follows from here that if $x^{(0)} \in \text{ri}S^4$, then $\alpha(x^{(0)}) \subset P$ and $\omega(x^{(0)}) \subset Q$. $\square$

Theorem 4.4. *The route of the positive trajectory is shown in $V_1 : T_2 \rightarrow T_1 \rightarrow Q$, and the route of the negative trajectory is shown in $V_1^{-1} : T_1 \rightarrow T_2 \rightarrow P$.*

Proof. According to Lemma 4.3. the intersection of polyhedra corresponding to the signatures δ_1 and δ_2 has the form $T_1 \cap T_2 = \text{co}\{e_1, A, B, C\}$ and for arbitrary point of the simplex, the following relations hold: $x_1' = x_1$, $x_2' < x_2$, $x_3' < x_3$, $x_4' < x_4$, $x_5' > x_5$. Then, for second-order iterations, we have:

$$\begin{aligned} x_1'' &= x_1' (ax_2' + bx_3' + cx_4' - dx_5') < x_1 (1 + ax_2 + bx_3 + cx_4 - dx_5) = x_1', \\ x_2'' &= x_2' (1 - ax_1') < x_2 (1 - ax_1) = x_2', \\ x_3'' &= x_3' (1 - bx_1') < x_3 (1 - bx_1) = x_3', \\ x_4'' &= x_4' (1 - cx_1') < x_4 (1 - cx_1) = x_4', \\ x_5'' &= x_5' (1 + dx_1') > x_5 (1 + dx_1) = x_5'. \end{aligned}$$

As a result, the relation $V_1(T_1 \cap T_2) \subset T_1$ is fulfilled. This means that the route will go from the T_2 to T_1 , that is, $T_2 \rightarrow T_1$. If you keep in mind that $Q \subset T_1$, then we have $V_1(Q \cap T_1) \subset Q$ and we will get the route $T_1 \rightarrow Q$. This means that the route of the positive trajectory looks like $V_1 : T_2 \rightarrow T_1 \rightarrow Q$. As we specified in the second part of the work, the mapping $V_1 : S^4 \rightarrow S^4$ is a homeomorphism. Therefore, for the inverse operator, that is, for the route of the negative trajectory we have $V_1^{-1} : T_1 \rightarrow T_2 \rightarrow P$. $\square$

Now let us move on to the representative of the second group. Let us assume that the skew-symmetric matrix we are given has the following form:

$$A_2 = \begin{pmatrix} 0 & a & b & -c & -d \\ -a & 0 & 0 & 0 & 0 \\ -b & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ where } 0 < a, b, c, d \leq 1. \quad (9)$$

Then the matrix A_2 defines a partially oriented bigraph (see Figure 4).

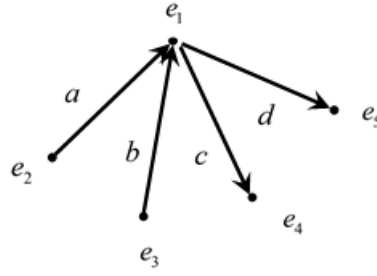


Figure 4: The partially oriented bigraph corresponding to the matrix A_2 .

In this case the operator $V_2 : S^4 \rightarrow S^4$, defined by the matrix A_2 , will have the following form:

$$V_2 : \begin{cases} x_1' = x_1(1 + ax_2 + bx_3 - cx_4 - dx_5), \\ x_2' = x_2(1 - ax_1), \\ x_3' = x_3(1 - bx_1), \\ x_4' = x_4(1 + cx_1), \\ x_5' = x_5(1 + dx_1). \end{cases} \quad (10)$$

Here the solution of the system $P = \{x \in S^4 : Ax \geq 0\}$ consists of the polynomial $P = \text{co}\{e_2, e_3, A, B, C, D\}$, and the solution of the system $Q = \{x \in S^4 : Ax \leq 0\}$ consists of the polynomial $Q = \text{co}\{e_4, e_5, A, B, C, D\}$ (see Figure 5).

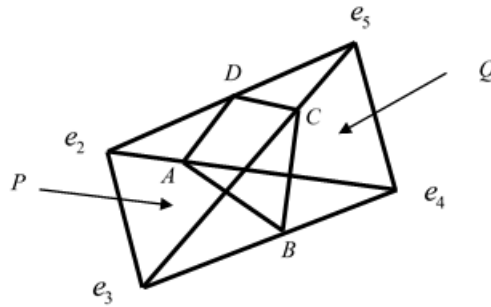


Figure 5: The sets of initial points of trajectories P and ending points of trajectories Q .

Here, the points $A \left(0, \frac{c}{a+c}, 0, \frac{a}{a+c}, 0\right)$, $B \left(0, 0, \frac{c}{b+c}, \frac{b}{b+c}, 0\right)$, $C \left(0, 0, \frac{d}{b+d}, 0, \frac{b}{b+d}\right)$, $D \left(0, \frac{d}{a+d}, 0, 0, \frac{a}{a+d}\right)$ consist of neutral points satisfying the equation $ax_2 + bx_3 - cx_4 - dx_5 = 0$ and lying in the same plane, and the polynomial consisting of the convex hull of these points, Γ_{ABCD} , also consists of neutral points. Solving the system $V_2x = x$, we find the fixed points of operator (10). We obtain the solution of this system in the form of the first vertex

$e_1 = (1, 0, 0, 0, 0) \in S^4$ and the face spanned by the vertices e_2, e_3, e_4 and e_5 , i.e.

$$\Gamma_{2345} = \left\{ (0, x_2, x_3, x_4, x_5) \in S^4 : 0 < x_2, x_3, x_4, x_5 \leq 1, \sum_{i=2}^5 x_i = 1 \right\}.$$

Here, as in the previous case, the vertex $e_1 = (1, 0, 0, 0, 0)$ is a saddle vertex.

Introduce the following notations (see Figure 5):

$$\Omega_3 = \text{co}\{e_2, e_3, A, B, C, D\} / \Gamma_{ABCD}, \quad \Omega_4 = \text{co}\{e_4, e_5, A, B, C, D\} / \Gamma_{ABCD}.$$

Theorem 4.5. *All points of the polyhedron from the set Ω_3 are repulsive, and all points from Ω_4 are attracting.*

Proof. The proof of this theorem is similar to the proof of Theorem 4.2. $\square$

The signs of the signature of the specified operator have the form $\delta_1 = (+ - - + +)$ and $\delta_2 = (- - - + +)$, and the solutions of the corresponding systems of inequalities are:

$$T_3 = \text{co}\{e_1, e_2, e_3, A, B, C, D\}, \quad T_4 = \text{co}\{e_1, e_4, e_5, A, B, C, D\}.$$

Lemma 4.6. *For sets T_3 and T_4 , the following holds: $V_2^{-1}(T_3) \subset T_3$ and $V_2(T_4) \subset T_4$.*

Proof. Show that $V_2^{-1}(T_3) \subset T_3$. Assume that $x' = V_2^{-1}x \in T_3$. In this case, the inequality $ax_2' + bx_3' - cx_4' - dx_5' \geq 0$ holds. This inequality and relations

$$\begin{aligned} ax_2' + bx_3' - cx_4' - dx_5' &= ax_2(1 - ax_1) + bx_3(1 - bx_1) - cx_4(1 + cx_1) - dx_5(1 + dx_1) = \\ &= ax_2 + bx_3 - cx_4 - dx_5 - (a^2x_1x_2 + b^2x_1x_3 + c^2x_1x_4 + d^2x_1x_5) \geq 0 \end{aligned}$$

imply the inequality $ax_2 + bx_3 - cx_4 - dx_5 \geq 0$. As a result, we have that $V_2^{-1}(T_3) \subset T_3$.

If $x' = V_2x \in T_4$, then we obtain the inequality $ax_2 + bx_3 + cx_4 - dx_5 \leq 0$. Therefore

$$\begin{aligned} ax_2' + bx_3' + cx_4' - dx_5' &= ax_2(1 - ax_1) + bx_3(1 - bx_1) + cx_4(1 - cx_1) - dx_5(1 + dx_1) = \\ &= ax_2 + bx_3 + cx_4 - dx_5 - (a^2x_1x_2 + b^2x_1x_3 + c^2x_1x_4 + d^2x_1x_5) \leq 0, \end{aligned}$$

what implies that $V_2(T_4) \subset T_4$.

Therefore, T_4 is invariant for a positive trajectory, and T_3 is invariant for a negative trajectory. We know that the intersection of polyhedra corresponding to the signatures δ_1 and δ_2 is

$$T_3 \cap T_4 = \text{co}\{e_1, A, B, C, D\}$$

and consists of a polyhedron, and for points $x \in T_3 \cap T_4$, the equality $x' = x$ holds. Then the sequence $\{x_1^{(n)}\}$, $n \in \mathbb{Z}$ reaches its single maximum value only in $T_3 \cap T_4$. Therefore, for an arbitrary interior point of the simplex $x^{(0)} \in \text{ri}S^4$, we get $\alpha(x^{(0)}) \subset P$ and $\omega(x^{(0)}) \subset Q$. $\square$

Theorem 4.7. *The route of the positive trajectory in $V_2 : T_3 \rightarrow T_4 \rightarrow Q$, and the route of the negative trajectory in $V_2^{-1} : T_4 \rightarrow T_3 \rightarrow P$.*

Theorem 4.7. is proved similarly to Theorem 4.4.

The remaining bigraphs belonging to the class $B_{1,4}$, corresponding to them skew-symmetric matrices and Lotka-Volterra systems are investigated in a similar way.

5. Conclusion

Mappings of type (1) were appeared in the nineties of the last century in the works Ganikhodzhaev (1993), and Ganikhodzhaev (1994). In these works, the connections of these mappings with the theory of tournaments and corresponding to them skew-symmetric matrices also were showed. These mappings are of interest for research because they can be used to study population genetics problems. In Eshmamatova (2023), Eshmamatova (2024b), Eshmamatova (2024a), Eshmamatova *et al.* (2023a), and Ganikhodzhaev *et al.* (2024) degenerate Lotka-Volterra mappings and corresponding to the partially oriented graphs of the third and fourth order are considered as discrete compartmental models for airborne viral diseases.

In this paper, quadratic Lotka-Volterra mappings acting in a four-dimensional simplex S^4 are considered, and bipartite graphs are associated with these mappings. Bigraphs are also divided into two subgroups based on isomorphy. They are split into groups: one vertex is in one group and the remaining four vertices are in another group. We consider one representative from each group, found the fixed points of the corresponding operators, and investigated their characters.

The study of bigraphs is motivated by the fact that they can be applied to many applied problems. For example, we can consider the problem of optimizing a railway schedule, where the input is a list of trains and their stops, and the goal is to minimize the number of stations visited by each train. This problem can be represented as a set of dominant vertices in a bigraph, with one vertex for each train, one for each station, and an edge between them if the train stops at the station.

Acknowledgement

We, the authors of the paper “Investigation of the asymptotic of quadratic Lotka-Volterra mappings and their connections with elements of graph theory” would like to thank the Tashkent State Transport University, as well as Universiti Teknologi MARA for the opportunity to publish our work.

References

- Amraoui S., Auroux D., Blum J. & Cosme E. 2023. Back-and-forth nudging for the quasi-geostrophic ocean dynamics with altimetry: Theoretical convergence study and numerical experiments with the future SWOT observations. *Discrete and Continuous Dynamical Systems-Series S* **16**(2): 197–219.
- Betounes D. 2010. *Differential Equations: Theory and Applications*. 2nd Ed. New York: Springer.
- Brauer F. & Castillo-Chávez C. 2001. *Mathematical Models in Population Biology and Epidemiology*. New York: Springer.
- Chartrand G., Lesniak L. & Zhang P. 2011. *Graphs and Digraphs*. 5th Ed. New York: CRC Press.
- Chau K.T. 2019. *Applications of Differential Equations in Engineering and Mechanics*. Florida: CRC Press.
- Cioba S.M. & Murty M.R. 2022. *A First Course in Graph Theory and Combinatorics*. 2nd Ed. Singapore: Springer.
- Devaney R.L. 2003. *An Introduction to Chaotic Dynamical Systems*. 2nd Ed. Florida: CRC Press.
- Devaney R.L. 2020. *A First Course in Chaotic Dynamical Systems: Theory and Experiment*. 2nd Ed. New York: CRC Press.
- Eshmamatova D., Tadzhieva M. & Ganikhodzhaev R. 2024. Invariant sets of Lotka-Volterra mappings acting in a four-dimensional simplex. *AIP Conference Proceedings* **3004**(1): 0020011.
- Eshmamatova D.B. 2023. Discrete analog of the SIR model. *AIP Conference Proceedings* **2781**(1): 020024.
- Eshmamatova D.B. 2024a. Compositions of Lotka-Volterra mappings as a model for the study of viral diseases. *AIP Conference Proceedings* **3085**(1): 020008.
- Eshmamatova D.B. 2024b. Dynamics of a discrete SIRD model based on Lotka-Volterra mappings. *AIP Conference Proceedings* **3004**(1): 020005.

- Eshmamatova D.B. & Ganikhodhzaev R.N. 2024. Asymptotic behavior of Volterra type discrete dynamical systems. *AIP Conference Proceedings* **3085**(1): 020009.
- Eshmamatova D.B., Ganikhodhzaev R.N. & Tadhzieva M.A. 2023a. Degenerate cases in Lotka-Volterra systems. *AIP Conference Proceedings* **2781**(1): 020034.
- Eshmamatova D.B., Tadhzieva M.A. & Ganikhodhzaev R.N. 2023b. Criteria for the existence of internal fixed points of Lotka-Volterra quadratic stochastic mappings with homogeneous tournaments acting in an $(m1)$ - dimensional simplex. *Journal of Applied Nonlinear Dynamics* **12**(4): 679–688.
- Eshmamatova D.B. & Yusupov F.A. 2024. Dynamics of compositions of some Lotka-Volterra mappings operating in a two-dimensional simplex. *Turkish Journal of Mathematics* **48**(3): 391–406.
- Ganikhodhzaev R., Eshmamatova D., Tadhzieva M. & Zakirov B. 2024. Some degenerate cases of discrete Lotka-Volterra dynamical systems and their applications in epidemiology. *Journal of Mathematical Sciences*. *Accepted*.
- Ganikhodhzaev R.N. 1993. Quadratic stochastic operators, Lyapunov functions and tournaments. *Sbornik: Mathematics* **76**(2): 489–506.
- Ganikhodhzaev R.N. 1994. A chart of fixed points and Lyapunov functions for a class of discrete dynamical systems. *Math. Notes* **56**(5): 1125–1131.
- Gantmacher F.R. 1960. *The Theory of Matrices*. New York: Chelsea Publishing Company.
- Griffin C.H. 2023. *Applied Graph Theory: An Introduction with Graph Optimization and Algebraic Graph Theory*. Singapore: World Scientific Publishing Co. Pte. Ltd.
- Harary F. & Palmer E.M. 1973. *Graphical Enumeration*. New York: Academic Press Inc.
- Holmgren R.A. 1996. *A First Course in Discrete Dynamical Systems*. 2nd Ed. New York: Springer.
- Hubbard J.H. & West B.H. 1995. *Differential Equations: A Dynamical Systems Approach: Higher-Dimensional Systems*. New York: Springer-Verlag.
- Ichiro T. 2001. Toward an interpretation of dynamic neural activity in terms of chaotic dynamical systems. *Behavioral and Brain Sciences* **24**(5): 793–847.
- Koh K.M., Dong F.M. & Tay E.G. 2007. *Introduction to Graph Theory: H3 Mathematics*. Singapore: World Scientific Publishing Co. Pte. Ltd.
- Martcheva M. 2015. *An Introduction to Mathematical Epidemiology*. New York: Springer.
- Oprocha P. & Wilczynski P. 2001. Shift spaces and distributional chaos. *Chaos, Solitons and Fractals* **31**(2): 347–355.
- Shapovalov S.A. 2000. A new solution of one Birkhoff problem. *Journal of Dynamical and Control Systems* **6**(3): 331–339.
- Strogatz S.H. 2015. *Nonlinear Dynamics and Chaos: With Application to Physics, Biology, Chemistry and Engineering*, 12, pp. 532. Taylor and Francis Group.
- Tadhzieva M.A., Eshmamatova D.B. & Ganikhodhzaev R.N. 2024. Volterra-type quadratic stochastic operators with a homogeneous tournament. *Journal of Mathematical Sciences* **278**(3): 546–556.
- Yang X., Ma R., Yang P. & Ban X.J. 2021. Link travel time estimation in double-queue-based traffic models. *Transport Engineering* **33**(3): 387–397.

Department of Mathematical Analysis and Differential Equations
Faculty of Physics and Mathematics
Fergana State University
Murabbiylar Street
Fergana, 705018
UZBEKISTAN
E-mail: ulugbek.muminov.2020@mail.ru

*Department of Higher Mathematics
Faculty of Civil Engineering
Tashkent State Transport University
Mirabad District, Temiryulchilar Street, 1
Tashkent, 100167
UZBEKISTAN
E-mail: 24dil@mail.ru, lolaxon@gmail.com*, Islamovyorqin72@gmail.com*

*Department of Mathematical Modeling of Nonlinear Systems
V.I. Romanovsky Institute of Mathematics of the Academy of Sciences of Uzbekistan
Olmazor District, University Street, 9
Tashkent, 100174
UZBEKISTAN
E-mail: 24dil@mail.ru*

Received: 11 October 2024

Accepted: 1 March 2025

*Corresponding author