

e^* —TRANSITIVITY AS AN EXTENSION TO CHAOS NOTIONS
(e^* —Transitiviti sebagai Suatu Perluasan kepada Konsep Kekalutan)

MALOUH BALOUSH, TAKASHI NOIRI & SYAHIDA CHE DZUL-KIFLI*

ABSTRACT

The introduction of e^* —transitivity is motivated by the broader concept of e^* —open sets, representing a generalised extension of various open set concepts. In this paper, we prove that the e^* —transitivity implies the classical transitivity and many other types of transitivity such as b —transitivity, δ —transitivity, α —transitivity, and sr —transitivity. Therefore, exploring the properties of e^* —transitivity holds significance, as these properties extend to other forms of transitivity. Additionally, we introduce several e^* —chaos notions that serve as analogues, to their classical counterparts. These notions include total e^* —transitivity, e^* —mixing, e^* —locally everywhere onto (abbreviated as e^* —l.e.o.), and weak e^* —blending. Subsequently, we demonstrate the interrelations among these newly introduced e^* —chaos notions and illustrate their connections to classical chaos concepts. This comparative analysis not only enriches our understanding of e^* —transitivity but also introduce new notions of chaos based on e^* —transitivity.

Keywords: e^* —transitivity; total e^* —transitivity; e^* —mixing; e^* —locally everywhere onto (e^* —l.e.o.)

ABSTRAK

Pengenalan e^* —transitiviti adalah didorong oleh konsep yang lebih luas iaitu set e^* —terbuka, yang mewakili satu pengitlakan umum daripada pelbagai konsep set terbuka. Dalam kertas ini, kami membuktikan bahawa e^* —transitiviti mengimplikasikan transitiviti klasik dan pelbagai jenis transitiviti lain seperti b —transitiviti, δ —transitiviti, α —transitiviti, dan sr —transitiviti. Oleh itu, penerokaan sifat-sifat e^* —transitiviti adalah penting kerana sifat-sifat ini diperluaskan ke jenis transitiviti yang lain. Tambahan pula, kami memperkenalkan beberapa konsep e^* —kalut yang berperanan sebagai analogi kepada konsep klasik, termasuk e^* —transitiviti menyeluruh, e^* —pencampuran, e^* —tempatan di mana-mana dan e^* —percampuran lemah. Seterusnya, kami menunjukkan hubungan antara konsep-konsep e^* —kalut yang baru diperkenalkan ini dan menjelaskan hubungannya dengan konsep kekalutan klasik. Analisis perbandingan ini bukan sahaja memperkayakan pemahaman kita tentang e^* —transitiviti tetapi juga memperkenalkan konsep kekacauan baharu berdasarkan e^* —transitiviti.

Kata kunci: e^* —transitiviti, e^* —transitiviti menyeluruh, e^* —pencampuran, e^* —tempatan di mana-mana (e^* —l.e.o.)

1. Introduction

The first use of the mathematical notion "chaos" was by Li and Yorke in 1975 (Li & Yorke 1975). Today there are various definitions of what it means for a map to be chaotic, most of which are based on the appropriate formalization of the stability concept and the notion of attractor. During the 1970s and 1980s, interest in chaotic dynamics increased rapidly, leading to various attempts to give the notion of chaos a mathematically precise meaning. However, chaos is not easy to define, and there is no universally agreed upon definition.

In 1989, Devaney introduced the concepts of topologically transitivity, density of periodic points and sensitive dependence on initial conditions. Using these properties, he defined what

is known as Devaney Chaos. There are many possible definitions of chaos in dynamical systems, some stronger and some weaker than Devaney's definition, such as distributional chaos (Schweizer & Smítal 1994), and topological chaos (Forti 2005). However, because of this particular definition can be applied to a large number of important examples and it is easy to verify, it become the most widely used definition of chaos. Until the late of 1980s, the study of chaotic dynamics was mainly restricted to research-oriented publications, particularly those by Devaney. His well-known definition of chaos state that for a continuous map $f : X \rightarrow X$, defined on a metric space X , the dynamical system (X, f) is said to be Devaney chaotic if it satisfies the following three conditions; f is (topologically) transitive, the periodic points of f are dense, and f exhibits sensitive dependence on initial conditions (Devaney 1989).

The condition of transitivity in Devaney's chaos has attracted many researchers as a fundamental aspects of chaos. Consequently, several alternative forms of transitivity has been introduced, including b -transitivity, δ -transitivity, α -transitivity, and sr -transitivity. Motivated by this, we define the e^* -transitive function using the e^* -open sets, having the way for a new notion of chaos to be introduced in the future based on e^* -transitivity.

In this paper, (X, τ) and (Y, σ) represent topological spaces. For a subset A of a space X , $Cl(A)$ and $Int(A)$ denote the closure and the interior of A , respectively. For a point x in X , $Orb_f(x) = \{x, f(x), f^2(x), \dots\}$ referred to briefly as the orbit.

2. Preliminaries and Definitions

In this section, we will revisit essential definitions and their properties, organised into three distinct parts: open sets and their extensions, transitivity and its extensions, and e^* -notion concepts.

2.1. Open sets and their extensions

Various extensions of open sets have been proposed to study different mathematical structures. These include b -open, b -regular, α -open, β -open, semiopen, e -open, etc. These extensions aim to capture and adapt the essence of open sets in diverse contexts beyond the classical topology. We list the definitions for these open sets below. Note that their closed sets are defined analogously as the complements of the respective open sets.

Definition 2.1. A subset A of a space (X, τ) is called

- (1) b -open (Andrijević 1996) if $A \subseteq Int(Cl(A)) \cup Cl(Int(A))$.
- (2) b -regular (Rajesh 2012) if it is b -open and b -closed.
- (3) α -open (Andrijević 1996) if $A \subseteq Int(Cl(Int(A)))$.
- (4) β -open (Abd El-Monsef *et al.* 1983) if $A \subseteq Cl(Int(Cl(A)))$.
- (5) Semiopen (Andrijević 1996) if $A \subseteq Cl(Int(A))$.
- (6) Regular open (Murad 2012a) if $A = Int(Cl(A))$.
- (7) δ -open (Veličko 1968) if for each $x \in A$ there exists a regular open set U such that $x \in U \subset A$.

Definition 2.2. (Murad 2012a) A space (X, τ) is called semi-regular space if its regular open sets form a basis for τ and its open set is called semi-regular open (or simply sr -open set).

For a general subset A , a point $x \in X$ is called a δ -cluster points of A if $A \cap Int(cl(U)) \neq \emptyset$ for each open set U containing x . The set of all δ -cluster points of A is called the δ -closure of A and is denoted by $Cl_\delta(A)$. The δ -interior of A is the union of all regular open sets contained in A and is denoted by $Int_\delta(A)$.

Definition 2.3. (Ekici 2008) A subset A of a space (X, τ) is said to be e -open if $A \subseteq Cl(Int_\delta(A)) \cup Int(cl_\delta(A))$.

2.2. Transitivity and its extension

The study of transitivity in chaos is essential for understanding the dynamic behavior of chaotic systems and the underlying structure of their trajectories. Transitivity is a property that characterizes the intermixing and complex dynamics within a system. In the study of a dynamical system, transitivity helps mathematicians and scientists analyze the long-term behavior of systems and make predictions about their trajectories. Given that we have introduced the concept of an open set and its extension, and transitivity is defined using open sets, it follows that there are many extensions of transitivity in literature review, i.e., sr –transitivity, b –transitivity, weakly b –transitivity, α –transitivity, and δ –transitivity. The following are the definitions of these types of transitivity.

Definition 2.4. (Kaki 2015) Let (X, τ) and (Y, σ) be two topological spaces, A function f is said to be sr –irresolute if for every sr –open subset H of Y , $f^{-1}(H)$ is sr –open in X . It is said to be semi-regular transitive (briefly: sr –transitive) if f is sr –irresolute and for every pair of non-empty sr –open sets U and V in X there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$.

The α – and b –irresolute functions, as well as their corresponding transitivity are defined in a similar manner, based on the respective open sets.

Definition 2.5. (Kaki 2013) Let (X, τ) be a topological space. A function $f : X \rightarrow X$ is said to be weakly b –irresolute if for each $x \in X$ and every b –open set V in X , there exists a b –open set U in X such that $f(U) \subset Cl_b(V)$, where $Cl_b(V)$ denotes the b –clouser of V .

Definition 2.6. (Kaki 2013) Let (X, τ) be a topological space and a function $f : X \rightarrow X$ be weakly b –irresolute. Then f is said to be topological weakly b –transitive if for every pair of non-empty b –regular open sets U and V in X there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$.

Definition 2.7. (Murad 2012b) A topological space (X, f) is said to be topologically δ –type transitive (or simply δ –transitive) if for every pair of nonempty δ –open sets U and V in X there is a nonnegative integer n such that $f^n(U) \cap V \neq \emptyset$.

2.3. e^* –notion in topology

In this section, we turn our focus towards the core of our investigation, the e^* –notion. The study of e^* –notions in topology offers a powerful framework for extending classical concepts such as open, closed, and dense sets. These generalised notions, based on e^* –open sets, enable us to explore the structure and behavior of topological spaces beyond traditional boundaries. In this subsection, we will examine how e^* –notion align with and extend classical topology, as well as discuss the relationships between e^* –open sets and other types of open sets.

Definition 2.8. (Ekici 2009) A subset A of a space X is called

- (1) e^* –open if $A \subset Cl(Int(Cl_\delta(A)))$, and
- (2) e^* –closed if $Int(Cl(Int_\delta(A))) \subset A$.

The e^* –closure of a set $A \subseteq X$ is intersection of all e^* –closed sets containing A and is denoted by $Cl_{e^*}(A)$. The e^* –interior of A is the union of all e^* –open sets contained in A and is denoted by $Int_{e^*}(A)$. Some authors prefer to use the notions δ – β –open instead of e^* –open set (Hatir & Noiri 2006).

Remark 2.9. Every e –open set is an e^* –open but the converse is not true.

Example 2.10. (Ekici 2009) Let $X = \{a, b, c, d\}$ and

$$\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$$

then the set $\{b, d\}$ is e^* -open, since $\{b, d\} \subset Cl(Int(Cl_\delta(\{b, d\})) = Cl(Int(\{a, b, c\})) = X$. But its not e -open because $Int_\delta(\{b, d\}) = \phi$, and $Cl(Int_\delta(\{b, d\}) \cup Int(Cl_\delta(b, d))) = Cl(\phi) \cup Int(\{a, b, c\}) = \phi \cup \{a, b, c\} = \{a, b, c\}$. Hence $\{b, d\} \not\subset \{a, b, c\}$. Therefore $\{b, d\}$ is not e -open set.

Theorem 2.11. (Ekici 2009) *The following properties hold true for any topological space;*

- (1) *The union of any family of e^* -open sets is an e^* -open set.*
- (2) *The intersection of any family of e^* -closed sets is an e^* -closed set.*

Therefore, the following lemma is also true.

Lemma 2.12. (Ekici 2009) *The following hold true for a subset A of a topological space (X, τ) :*

- (1) *$Int_{e^*}(A)$ is e^* -open,*
- (2) *$Cl_{e^*}(A)$ is e^* -closed,*
- (3) *$X \setminus Cl_{e^*}(A) = Int_{e^*}(X \setminus A)$.*

The following remark and examples show how e^* -open sets deviate from classical open set concepts.

Remark 2.13.

- (1) The intersection of two e^* -open sets is not necessarily e^* -open. Therefore the e^* -open sets in a space X do not form a topology.
- (2) The union of two e^* -closed sets is not necessarily e^* -closed.

The following counterexamples serve to further demonstrate and validate the distinctions between e^* -open sets and traditional open sets, as outlined in the previous remark.

Example 2.14. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then $U = \{a, c\}$ and $V = \{b, c\}$ are both e^* -open sets, but $U \cap V = \{c\}$ is not e^* -open.

Example 2.15. Continuing from Example 2.14, where $U = \{a, c\}$ and $V = \{b, c\}$ are both e^* -open sets. So the complement of U and V are $U^c = \{b\}$, $V^c = \{a\}$, respectively, which are e^* -closed sets. But $U^c \cup V^c = \{a, b\}$ is not e^* -closed.

However, some topological properties of e^* -openness remain similar to the classical concept of openness, as given as follows:

Proposition 2.16. (Hatir & Noiri 2009) *For the e^* -closure of a set A in a space X , the following properties hold:*

- (1) *A is e^* -closed in X if and only if $A = Cl_{e^*}(A)$,*
- (2) *$Cl_{e^*}(A) \subset Cl_{e^*}(B)$ whenever $A \subset B \subset X$,*
- (3) *$Cl_{e^*}(A)$ is e^* -closed in X ,*
- (4) *$Cl_{e^*}(Cl_{e^*}(A)) = Cl_{e^*}(A)$,*
- (5) *$x \in Cl_{e^*}(A)$ if $A \cap U \neq \phi$, for every e^* -open set U containing x .*

Lemma 2.17. (Hatir & Noiri 2009) *Let A be a subset of a space X . Then*

- (1) *$Cl_\delta(A) \cap V \subset Cl_\delta(A \cap V)$, for any δ -open set V in X ,*
- (2) *$Int_\delta(A \cup F) \subset Int_\delta(A) \cup F$, for any δ -closed set F in X .*

Proposition 2.18. (Hatir & Noiri 2009) *A set A in X is e^* -open if and only if $U \cap A$ is e^* -open in X , for every regular open set U of X .*

Next, we examine how the concepts of interior and closure in e^* -notions relate to the classical topological notions of interior and closure.

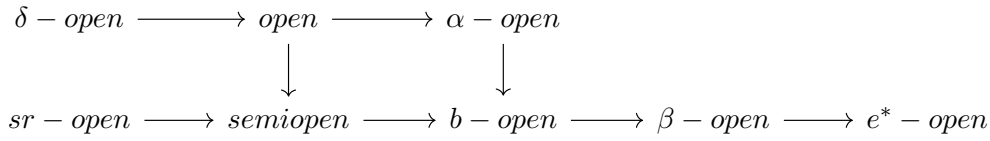


Figure 1: Relationship between different types of open sets

Lemma 2.19. (Hatir & Noiri 2009) *Let A be a subset of a topological space (X, τ) , then*

- (1) $\text{Int}_{e^*}(A) = A \cap \text{Cl}(\text{Int}(\text{Cl}_\delta(A)))$,
- (2) $\text{Cl}_{e^*}(A) = A \cup \text{Int}(\text{Cl}(\text{Int}_\delta(A)))$.

From the above definitions, the relationship between the different types of open sets can be summarised in Figure 1. Also, see (Farhan & Yang 2015).

We now introduce a new topological term within the framework e^* –notions.

Definition 2.20. A subset A of a space (X, τ) is said to be

- (1) e^* –dense if $\text{Cl}_{e^*}(A) = X$,
- (2) nowhere e^* –dense if $\text{Int}_{e^*}(\text{Cl}_{e^*}(A)) = \phi$.

Proposition 2.21. Any e^* –dense subset in X intersects any nonempty e^* –open set in X .

Proof. It is obvious by definition. $\square$

The following lemma will be useful later in this paper.

Lemma 2.22. (Hatir & Noiri 2009) *If U is e^* –open in X , and V is e^* –open in Y , then $U \times V$ is e^* –open in $X \times Y$.*

3. Properties and Dynamics of e^* –Transitive Function

In the third section, we present our findings on e^* –transitivity. Before delving into the dynamics of e^* –transitive maps, we will first investigate the properties of e^* –continuous functions, and introduce the concepts of e^* –open functions and e^* –irresolute functions. Following this, we will explore the relationships between various types of transitivity and e^* –transitivity, demonstrating that e^* –transitivity implies all other forms of transitivity.

3.1. Characterization of e^* –continuous function

In the context of topology, the notions of continuity and resoluteness are fundamental in understanding the behaviour of functions between topological spaces.

Definition 3.1. (Veličko 1968) A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called e^* –continuous (or δ – β –continuous (Hatir & Noiri 2006)) if the preimage of every open set A in Y is an e^* –open set in X .

Definition 3.2. (Ekici 2009) A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called e^* –irresolute if the inverse image of each e^* –open set in Y is e^* –open in X .

Building on these foundational definitions, we now introduce the concept of e^* –continuous functions, which will advance our exploration of e^* –transitivity and its implications in topology.

Definition 3.3. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be e^* -open if for every open set $A \in \tau$, $f(A)$ is e^* -open in Y .

Proposition 3.4. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions. Then, the following properties hold:

- (1) If f is e^* -continuous and g is continuous, then $g \circ f$ is e^* -continuous.
- (2) If f is e^* -irresolute and g is e^* -continuous, then $g \circ f$ is e^* -continuous.
- (3) If f is onto e^* -open, and $g \circ f : X \rightarrow Z$ is continuous, then g is e^* -continuous.
- (4) If f is onto continuous and $g \circ f : X \rightarrow Z$ is e^* -open, then g is e^* -open.
- (5) If g is an e^* -continuous injection and $g \circ f : X \rightarrow Z$ is open, then f is e^* -open.
- (6) If $h : X \rightarrow X \times Y$ defined by $h(x) = (x, f(x))$ for every $x \in X$ is e^* -continuous, then f is e^* -continuous.

Proof. (1) Let f and g be e^* -continuous and continuous functions, respectively, and let U be a nonempty open set in Z . Since g is continuous, $g^{-1}(U)$ is open in Y . Since f is e^* -continuous, $f^{-1}(g^{-1}(U))$ is e^* -open in X . But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. Hence $g \circ f$ is e^* -continuous.

(2) Let f and g be e^* -irresolute and e^* -continuous, respectively, and let U be a nonempty open set in Z . Since g is e^* -continuous, $g^{-1}(U)$ is e^* -open in Y . Since f is e^* -irresolute, $f^{-1}(g^{-1}(U))$ is e^* -open in X . But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. Hence $g \circ f$ is an e^* -continuous function.

(3) Let U be a nonempty open set in Z . Since $g \circ f : X \rightarrow Z$ is continuous, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in X . Since f is onto e^* -open, $f(f^{-1}(g^{-1}(U))) = g^{-1}(U)$ is e^* -open in Y . Hence g is e^* -continuous.

(4) Let U be a nonempty open set in Y . Since f is continuous, $f^{-1}(U)$ is open in X . But $g \circ f : X \rightarrow Z$ is e^* -open, then $(g \circ f)(f^{-1}(U)) = g(f(f^{-1}(U)))$ is e^* -open in Z . Since f is onto, $g(f(f^{-1}(U))) = g(U)$ is e^* -open in Z . Hence g is e^* -open.

(5) Let U be a nonempty open set in X . Since $g \circ f : X \rightarrow Z$ is open, $g(f(U))$ is open in Z . Therefore, $g^{-1}(g(f(U))) = f(U)$ is e^* -open in Y because g is e^* -continuous and injective. Hence f is e^* -open.

(6) Let h be an e^* -continuous function, and let U be a nonempty open set in Y , then $X \times U$ is open in $X \times Y$. Since h is e^* -continuous, it follows that $h^{-1}(X \times U) = f^{-1}(U)$ is e^* -open in X , and therefore f is e^* -continuous. $\square$

3.2. The dynamics of e^* -transitive maps

In this section, we explore the dynamics of e^* -transitive maps and examine their relationships with the various types of transitivity discussed earlier. We will demonstrate that e^* -transitivity serves as a foundational concept, implying all other forms of transitivity. Through this analysis, we aim to highlight the significance of e^* -transitivity within the broader landscape of topological properties.

Definition 3.5. Let (X, τ) be a topological space and let $x \in X$. A subset U of X is said to be an e^* -neighbourhood of x if there exists an e^* -open set G such that $x \in G \subset U$.

Definition 3.6. Let $f : X \rightarrow X$ be an e^* -irresolute function, then a point $x \in X$ is called an e^* -non-wandering point if for any e^* -neighbourhood U of x there exists an integer $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$. The set of e^* -non-wandering points is denoted by $\Omega_{e^*}(f)$.

Proposition 3.7. Let $f : X \rightarrow X$ be an e^* –irresolute function. For the set $\Omega_{e^*}(f)$, we have;

- (1) $\Omega_{e^*}(f)$ is e^* –closed.
- (2) $\Omega_{e^*}(f)$ is f –invariant.

Proof. (1) To see that $\Omega_{e^*}(f)$ is e^* –closed, we show its complement is e^* –open. If $x \notin \Omega_{e^*}(f)$, then there exists an e^* –neighbourhood U of x such that $f^n(U) \cap U = \emptyset$ for all $n \geq 1$. So, there exists an e^* –open set $V_x \subset U$ containing x . Then all points $y \in V_x$ also do not belong to $\Omega_{e^*}(f)$. Thus the complement of $\Omega_{e^*}(f)$, since the union of e^* –open sets V_x over all $x \notin \Omega_{e^*}(f)$ is indeed e^* –open by Theorem 2.11. is e^* –open.

(2) To show $\Omega_{e^*}(f)$ is f –invariant, let $x \in \Omega_{e^*}(f)$. Let V denote an e^* –neighbourhood of $f(x)$. Then $U = f^{-1}(V)$ is an e^* –neighbourhood of x , and hence there exists some integer $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$. The image of this intersection under f is contained in $f^n(V) \cap V$, and hence $f^n(V) \cap V \neq \emptyset$. Thus $f(x) \in \Omega_{e^*}(f)$. *square*

Definition 3.8. Let X be a topological space and f be an e^* –irresolute function on X . Then (X, f) is called e^* –minimal system if X does not contain any non-empty, proper, e^* –closed f –invariant subset. Furthermore, it is equivalent to either one of the following:

- (1) The orbit of each point of X is e^* –dense in X .
- (2) $Cl_{e^*}(Orb_f(x)) = X$ for each $x \in X$.
- (3) For $x \in X$ and a nonempty e^* –open set U in X , there exists $n \in \mathbb{N}$ such that $f^n(x) \in U$.

Theorem 3.9. Let $f : X \rightarrow X$ be an e^* –irresolute function, then the following are equivalent:

- (1) f is an e^* –minimal map,
- (2) For a nonempty e^* –open set U of X , $X = \bigcup_{n=0}^{\infty} f^{-n}(U)$.

Proof. (1) $\Rightarrow$ (2): Let U be any nonempty e^* –open set of X . By (1), for each $x \in x$, there exists $n \in \mathbb{N}$ such that $f^n(x) \in U$. Thus $x \in f^{-n}(U)$ and hence $X \subset \bigcup_{n=0}^{\infty} f^{-n}(U)$. Therefore, $X = \bigcup_{n=0}^{\infty} f^{-n}(U)$.

(2) $\Rightarrow$ (1): Let U be any non empty e^* –open set of X , and let $x \in X$. Since $x \in X = \bigcup_{n=0}^{\infty} f^{-n}(U)$, $x \in f^{-n}(U)$ for some $n > 0$. Hence $f^n(x) \in U$, i.e. the orbit of every point x in X is e^* –dense in X . Therefore, f is an e^* –minimal function. $\square$

Definition 3.10. Let X be a topological space and $f : X \rightarrow X$ be an e^* –irresolute function. The function $f : X \rightarrow X$ is said to be e^* –transitive if for any non-empty e^* –open subsets $U, V \subset X$, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$.

Proposition 3.11. For an e^* –irresolute function $f : X \rightarrow X$, the following properties are equivalent:

- (1) f is e^* –transitive;
- (2) For every nonempty e^* –open set U in X , $\bigcup_{n=0}^{\infty} f^n(U)$ is e^* –dense in X ;
- (3) For every nonempty e^* –open set U in X , $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is e^* –dense in X ;

(4) If B is e^* -closed and $f(B) \subset B$, then $B = X$ or B is nowhere e^* -dense in X ;

(5) If U is e^* -open and $f^{-1}(U) \subset U$, then $U = \emptyset$ or U is e^* -dense in X .

Proof. (1) $\Rightarrow$ (2): Let f be e^* -transitive, and let U be a nonempty e^* -open set in X . Suppose that $\bigcup_{n=0}^{\infty} f^n(U)$ is not e^* -dense in X , then there exists an e^* -open set V such that $\bigcup_{n=0}^{\infty} f^n(U) \cap V = \phi$. This means that $f^n(U) \cap V = \phi$ for all $n \in \mathbb{N}$, which is a contradiction since f is e^* -transitive.

(2) $\Rightarrow$ (3): Suppose that there exists a nonempty e^* -open set U in X such that $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is not e^* -dense in X . Then, there exists a nonempty e^* -open set V such that $\bigcup_{n=0}^{\infty} f^{-n}(U) \cap V = \phi$ and $f^{-n}(U) \cap V = \phi$ for every n and $U \cap f^n(V) = \phi$. Therefore, $U \cap (\bigcup_{n=0}^{\infty} f^n(V)) = \phi$ and $\bigcup_{n=0}^{\infty} f^n(V)$ is not e^* -dense.

(3) $\Rightarrow$ (1): Let U and V be any nonempty e^* -open sets. By (3), $\bigcup_{n=0}^{\infty} f^{-n}(V)$ is e^* -dense in X and $U \cap \bigcup_{n=0}^{\infty} f^{-n}(V) \neq \phi$. Hence, there exists a positive integer m such that $U \cap f^{-m}(V) \neq \phi$. Therefore, $\phi \neq f^m(U \cap f^{-m}(V)) \subset f^m(U) \cap V$. This shows that f is e^* -transitive.

(1) $\Rightarrow$ (4): Let B be e^* -closed and $f(B) \subset B$. Suppose that $B \neq X$ and B is not nowhere e^* -dense. Let $V = X \setminus B$. Then V is a nonempty e^* -open set. Let $W = \text{Int}_{e^*}(B)$. Then W is a nonempty e^* -open set and $W \subset B$. Hence $f(W) \subset f(B) \subset B$ and $f^n(W) \subset B$. Therefore, $f^n(W) \cap V \subset B \cap (X \setminus B) = \phi$ for every positive integer n . This is contrary that f is e^* -transitive. Thus, $B = X$ or B is nowhere e^* -dense.

(4) $\Rightarrow$ (5): Suppose that U is an e^* -open set and $f^{-1}(U) \subset U$. Let $B = X \setminus U$. Then B is e^* -closed and $f^{-1}(X \setminus B) \subset X \setminus B$. Hence $X \setminus f^{-1}(B) \subset X \setminus B$ and $B \subset f^{-1}(B)$. Therefore, $f(B) \subset B$. By (4), $B = X$ or B is nowhere e^* -dense. If $B = X$, then $U = \phi$. If $B = X \setminus U$ is nowhere e^* -dense, then $\phi = \text{Int}_{e^*}(Cl_{e^*}(B)) = \text{Int}_{e^*}(Cl_{e^*}(X \setminus U)) = \text{Int}_{e^*}(X \setminus \text{Int}_{e^*}(U)) = \text{Int}_{e^*}(X \setminus U) = X \setminus Cl_{e^*}(U)$. Therefore, U is e^* -dense in X .

(5) $\Rightarrow$ (3): Let U be any nonempty e^* -open set and let $V = \bigcup_{n=0}^{\infty} f^{-n}(U)$. Then, V is a nonempty e^* -open set. Since $f^{-1}(V) \subset f^{-1}(V) \cup U \subset V$, by (5) V is e^* -dense and $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is e^* -dense. $\square$

So, for e^* -transitivity we state the following theorem.

Theorem 3.12. *Let f be an e^* -transitive function, then the following hold*

- (1) f is transitive.
- (2) f is b -transitive.
- (3) f is δ -transitive.

- (4) f is α –transitive.
- (5) f is sr –transitive.

Proof. (1) Let U and V be two nonempty open sets in X , and let $f : X \rightarrow X$ be an e^* –transitive map. Since U and V are nonempty open sets, U and V are nonempty e^* –open sets since every open set is e^* –open. By the e^* –transitivity of f , there exists an integer $n > 0$ such that $f^n(U) \cap V \neq \emptyset$. Hence f is transitive. The other implications follow the same argument in (1). $\square$

Corollary 3.13. *Let f be an e^* –transitive function, then f is weakly b –transitive.*

Proof. This is obvious since every b –transitive is weakly b –transitive (Kaki 2013). $\square$

Proposition 3.14. Let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two functions. If $f \times g$ is e^* –transitive, then f and g are both e^* –transitive.

Proof. Let $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$ be e^* –open sets in X, Y , respectively. Then by Lemma 2.22, $U_1 \times V_1$ and $U_2 \times V_2$ are e^* –open sets in $X \times Y$. Since $f \times g$ is e^* –transitive, there exists an integer $n > 0$, we have;

$$\begin{aligned} \phi &\neq (f \times g)^n(U_1 \times V_1) \cap (U_2 \times V_2) \\ &= (f^n(U_1) \times g^n(V_1)) \cap (U_2 \times V_2) \\ &= (f^n(U_1) \cap U_2) \times (g^n(V_1) \cap V_2) \end{aligned}$$

Hence, $f^n(U_1) \cap U_2 \neq \emptyset$ and $g^n(V_1) \cap V_2 \neq \emptyset$. Therefore, f and g both are e^* –transitive. $\square$

Now, we introduce the definitions of total e^* –transitivity, e^* –mixing, e^* –locally everywhere onto, and weak e^* –blending. We will prove that these concepts imply their classical counterparts: total transitivity, mixing, locally everywhere onto, and weak blending, respectively. Additionally, we establish that e^* –l.e.o. implies e^* –transitivity, total e^* –transitivity, e^* –mixing, and weak e^* –blending.

Definition 3.15. Let X be a topological space and $f : X \rightarrow X$ be an e^* –irresolute function, then f is said to be

- (1) Total e^* –transitivity if f^n is e^* –transitive for all integers $n \geq 1$.
- (2) Topologically e^* –mixing if whenever U and V are nonempty e^* –open subsets of X , there exists $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$.
- (3) e^* –locally everywhere onto or simply e^* –l.e.o. if for every e^* –open subset $U \subseteq X$ there exists a positive integer n such that $f^n(U) = X$.
- (4) Weak e^* –blending if for any pair of non-empty e^* –open sets U and V in X , there is some integer $n > 0$ such that $f^n(U) \cap f^n(V) \neq \emptyset$.

Lemma 3.16. *If f is e^* –l.e.o., then it is l.e.o. Similar implications hold for topological e^* –mixing, total e^* –transitivity, and weak e^* –blending towards their classical counterparts.*

Proof. let U be any nonempty open subset of X . Since every open set is e^* –open, and since f is e^* –l.e.o., then there exists an integer $n > 0$ such that $f^n(U) = X$. Therefore, f is l.e.o. The implications for other e^* –notions are proved similarly by using their definitions and the fact that every open set is e^* –open. $\square$

Proposition 3.17. For any e^* –irresolute function $f : X \rightarrow X$, the following implications hold:

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), where

- (1) e^* –l.e.o.,
- (2) topological e^* –mixing,

- (3) total e^* –transitivity,
- (4) e^* –transitivity.

Proof. (1) $\Rightarrow$ (2): Let U, V be any two nonempty e^* –open sets in X . Since f is e^* –l.e.o., there exists a positive integer n such that for every e^* –open set U , $f^n(U) = X$, and then $f^n(U) \cap V \neq \emptyset$. So we can choose $N > n$ such that $f^k(U) \cap V \neq \emptyset$ for every $k \geq N$. Hence f is topologically e^* –mixing.

(2) $\Rightarrow$ (3): Let f be topologically e^* –mixing and let U, V be nonempty e^* –open sets. There exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $f^n(U) \cap V \neq \emptyset$. To see that f^m is e^* –transitive for every integer m , let M be the smallest multiple of m which is greater than N . Therefore $f^M(U) \cap V \neq \emptyset$, and hence f^m is e^* –transitive.

(3) $\Rightarrow$ (4): This obvious by the definition of total e^* –transitivity. $\square$

From the previous findings, the following diagram (Figure 2) holds:

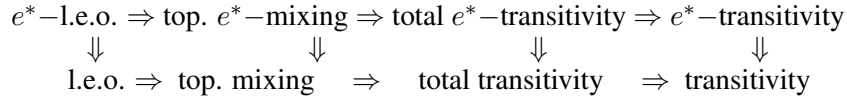


Figure 2: Relationship between e^* –notions and their classical counterparts

Proposition 3.18. If $f : X \rightarrow X$ is e^* –l.e.o., then f is weak e^* –blending.

Proof. Let U, V be any two nonempty e^* –open sets in X . Since f is e^* –l.e.o., then there exist $n_1, n_2 > 0$ such that $f^{n_1}(U) = f^{n_2}(V) = X$. Without the lose of generality, let $n_1 > n_2$. Then $f^{n_1}(U) = f^{n_1}(V) = X$ and hence, $f^{n_1}(U) \cap f^{n_1}(V) = X \neq \emptyset$. Therefore, f is weak e^* –blending. $\square$

Proposition 3.19. Let $f : X \rightarrow X$ and $g : X \rightarrow X$ be e^* –irresolute functions.

- (1) If $f \times g$ is topologically e^* –mixing, then f and g are topologically e^* –mixing,
- (2) If $f \times g$ is e^* –l.e.o., then f and g are e^* –l.e.o.,
- (3) If $f \times g$ is total e^* –transitive, then f and g are total e^* –transitive.

Proof. (1) Let $f \times g$ be topologically e^* –mixing. Let U_1, U_2 and V_1, V_2 be e^* –open sets in X and Y , respectively. By Lemma 2.17, $U_1 \times V_1$ and $U_2 \times V_2$ are e^* –open sets in $X \times Y$. Since $f \times g$ is topologically e^* –mixing, there exists an $N \in \mathbb{N}$ such that $(f \times g)^n(U_1 \times V_1) \cap (U_2 \times V_2) \neq \emptyset$ for every $n > N$. Similarly with Proof of Proposition 3.18, we have $(f \times g)^n(U_1 \times V_1) \cap (U_2 \times V_2) = (f^n(U_1) \cap U_2) \times (g^n(V_1) \cap V_2) \neq \emptyset$. Therefore $f^n(U_1) \cap U_2 \neq \emptyset$ and $g^n(V_1) \cap V_2 \neq \emptyset$ for every $n > N$. Therefore, f and g are topologically e^* –mixing.

(2) Let $f \times g$ be e^* –l.e.o. Let U and V be e^* –open sets in X and Y , respectively. By Lemma 2.17, $U \times V$ is an e^* –open set in $X \times Y$. Since $f \times g$ is e^* –l.e.o., there exists an $n \in \mathbb{N}$ such that $(f \times g)^n(U \times V) = f^n(U) \times g^n(V) = X \times Y$. Therefore, $f^n(U) = X$ and $g^n(V) = Y$. This shows that f and g are e^* –l.e.o.

(3) Suppose that $f \times g$ is totally e^* –transitive. Then $(f \times g)^n$ is e^* –transitive for each $n \in \mathbb{N}$. For any e^* –open sets U_1, U_2 and V_1, V_2 in X and Y , respectively, $U_1 \times V_1$ and $U_2 \times V_2$ are e^* –open sets in $X \times Y$. Hence for any $n \in \mathbb{N}$ there exists an $m(n) \in \mathbb{N}$ such that $((f \times g)^n)^{m(n)}(U_1 \times V_1) \cap (U_2 \times V_2) \neq \emptyset$. Therefore, we obtain $(f^n)^{m(n)}(U_1) \cap U_2 \neq \emptyset$ and $(g^n)^{m(n)}(V_1) \cap V_2 \neq \emptyset$ for an $m(n)$ and for every $n \in \mathbb{N}$. Hence, f^n and g^n are e^* –transitive for any $n \in \mathbb{N}$. This shows that f and g are totally e^* –transitive. $\square$

4. Conclusion

In this paper, we introduced the concept of e^* –transitivity and explored its properties, as well as its relationships with various other types of transitivity. We demonstrated that e^* –transitivity implies classical transitivity and several other forms, including b –transitivity, δ –transitivity, α –transitivity, and sr –transitivity. Furthermore, we defined the concepts of total e^* –transitivity, e^* –mixing, e^* –l.e.o., and weak e^* –blending. We proved that these e^* –concepts imply their classical counterparts: total transitivity, mixing, l.e.o., and weak blending, respectively. Additionally, we established that e^* –l.e.o. further implies e^* –transitivity, total e^* –transitivity, e^* –mixing, and weak e^* –blending.

Acknowledgement

The authors would like to thank referee sincerely for very helpful comments improving the paper.

References

- Abd El-Monsef M., El-Deeb S. & Mahmoud R. 1983. β –open sets and β –continuous mapping. *Bulletin of the Faculty of Science, Assiut University*, **12**: 77–90.
- Andrijević D. 1996. On b –open sets. *Matematički Vesnik* **48**: 59–64.
- Devaney R. 1989. *An Introduction to Chaotic Dynamical Systems*. 2nd Ed. Massachusetts, USA: Addison-Wesley Publishing Company.
- Ekici E. 2008. On e –open sets, dp^* –sets and dpe^* –sets and decompositions of continuity. *Arabian J. Sci. Eng.* **33**: 269–282.
- Ekici E. 2009. On e^* –open sets and $(d, s)^*$ –sets. *Mathematica Moravica* **13**: 29–36.
- Farhan A.M. & Yang X.S. 2015. New type of strongly continuous functions in topological spaces via $\delta - \beta$ –open sets. *European Journal of Pure and Applied Mathematics* **8**(2): 185–200.
- Forti G.L. 2005. Various notions of chaos for discrete dynamical systems: A brief survey. *Aequationes Mathematicae* **70**(1-2): 1–13.
- Hatir E. & Noiri T. 2006. Decompositions of continuity and complete continuity. *Acta Mathematica Hungarica* **113**(4): 281–287.
- Hatir E. & Noiri T. 2009. On $\delta - \beta$ –continuous functions. *Chaos, Solitons and Fractals* **42**(1): 205–211.
- Kaki M.N.M. 2013. Introduction to weakly b –transitive maps on topological spaces. *Sci. Res.* **1**(4): 59–62.
- Kaki M.N.M. 2015. Mixing, weakly mixing and transitive sets. *International Research Journal of Engineering and Technology* **2**(8): 1596–1607.
- Li T.J. & Yorke J.A. 1975. Period three implies chaos. *Amer. Math. Monthly* **82**(10): 985–992.
- Murad M.N. 2012a. New types of transitive functions and minimal systems. *Int. J. Basic Appl. Sci.* **12**(1): 53–58.
- Murad M.N. 2012b. Topologically α –transitive maps and minimal systems. *General Mathematics Notes* **10**(2): 43–53.
- Rajesh N. 2012. On weakly b –irresolute functions. *Acta Univ. Apulensis* **30**: 153–160.
- Schweizer B. & Smítal J. 1994. Measures of chaos and a spectral decomposition of dynamical systems on the interval. *Transactions of the American Mathematical Society* **344**(2): 737–754.
- Veličko N.V. 1968. H –closed topological spaces. *Amer. Math. Soc. Transl.* **2**: 103–118.

Department of Basic Sciences,
National University College of Technology
Amman, 11621, JORDAN
E-mail: malohblosh@yahoo.com

*222949-1 Shiokita-cho, Hinagu
Yatsushiro-shi Kumamoto-ken
869-5142, JAPAN
E-mail: t.noiri@nifty.com*

*Department of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 UKM Bangi
Selangor, MALAYSIA
E-mail: syahida@ukm.edu.my**

Received: 16 October 2024

Accepted: 1 March 2025

*Corresponding author