

A NEW MODIFIED COMPACT FINITE DIFFERENCE SCHEME FOR SOLVING THE GENERALIZED BURGERS-HUXLEY EQUATION IN TEACHING AND LEARNING PROCESSES

(Skema Diubah Suai Baru Beza Terhingga Padat untuk Menyelesaikan Persamaan Burgers-Huxley Teritlak dalam Proses Pengajaran dan Pembelajaran)

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ABSTRACT

In this study, a new modified scheme based on compact finite difference technique has been proposed for approximate the solution of the generalized Burgers-Huxley equation. The basic idea of compact finite difference scheme is to find the compact approximations to the derivatives terms by manipulating the governing equation. This basic concept is suitable to be introduced in teaching and learning processes for the Generalized Burgers – Huxley equation. The numerical experiments on the proposed scheme have been conducted and all the computations were carried out by MATLAB R2010a software. The results are compared with analytical solutions and the solution obtained by other researchers. Encouraging results have been obtained. Comparative study shows that the compact finite difference scheme produces better accuracy than the scheme cited in the literature. The compact finite difference scheme exhibits reliable method for solving the generalized Burgers-Huxley equation.

Keywords: compact finite difference scheme; generalized Burgers-Huxley equation; accuracy

ABSTRAK

Dalam kajian ini, satu skema diubah suai baru yang berdasarkan teknik beza terhingga padat telah dicadangkan untuk mendapatkan penghampiran penyelesaian bagi persamaan Burgers-Huxley teritlak. Idea asas skema beza terhingga padat adalah untuk mencari penghampiran padat bagi terma-terma pembezaan dengan memanipulasi persamaan pengawal. Konsep asas ini sesuai diperkenalkan di dalam proses pengajaran dan pembelajaran untuk persamaan Burgers-Huxley teritlak. Ujikaji berangka ke atas skema yang dicadangkan telah dijalankan dan semua pengiraan dilakukan menggunakan perisian MATLAB R2010a. Hasil yang diperolehi dibandingkan dengan penyelesaian analitik dan penyelesaian yang diperolehi oleh penyelidik lain. Hasil yang menggalakkan telah diperolehi. Kajian perbandingan menunjukkan bahawa skema beza terhingga padat menghasilkan ketepatan yang lebih baik berbanding skema didalam kajian literatur. Skema beza terhingga padat menunjukkan kaedah yang boleh dipercayai untuk menyelesaikan persamaan Burgers-Huxley teritlak.

Kata kunci: skema beza terhingga padat; persamaan Burgers-Huxley teritlak; kejituan

1. Introduction

The generalized Burgers-Huxley equation represents the most comprehensive version of the Huxley equation. It is applied across various fields, including chemistry, biology, combustion, metallurgy, engineering and mathematics.

In recent years, the Burgers-Huxley equation has been solved by various powerful numerical approximations: Finite element approximation for a delayed equation with weakly

singular kernels (Mahajan *et al.* 2024). An accurate and efficient numerical solution for the equation via Taylor wavelets method (Korkut 2023), Numerical solution of equation using Deep Galerkin neural network method (Kumar *et al.* 2022), Pseudospectral method and Darvishi's preconditioning (Javidi 2006), variational iteration method (VIM) (Batiha *et al.* 2008), high-order finite difference schemes (Sari *et al.* 2011), Iterative Differential Quadrature (IDQ) (Tomasiello 2010), Haar wavelet method (Çelik 2012), local discontinuous Galerkin method (Zhang *et al.* 2012), hyperbolic tangent method (Krisnangkura *et al.* 2012), B-spline collocation method (Mohammadi 2013), the Laplace decomposition method (Yaghouti & Zabihi 2013), and Cardano's method (Macías-Díaz 2014).

According to Spotz and Carey (1995), the basic approach in the development of a high-order compact differencing scheme is utilizing the governing differential equations to help approximate the derivative terms in truncation errors. By including the compact approximations to the leading truncation error terms, the high-order compact finite schemes increase the accuracy of the classical difference approximation (Spotz 1995).

A high-order compact finite difference technique has been applied to solve various differential equations including the nonlinear Black-Scholes equation (Düring *et al.* 2003), second-order integro-differential equations (Zhao & Corless 2006), two-dimensional second-order elliptic equations (Zhao *et al.* 2006), the generalized Burgers-Fisher equation (Sari *et al.* 2011), the Burgers equation (Liao 2008), (Sari & Gürarlan 2009), two-dimensional reaction – diffusion elliptic singularly-perturbed problems (Gracia & Clavero 2006), one-dimensional fractional diffusion equation (Cui 2009), the Helmholtz equation (Feng *et al.* 2011), two-dimensional fractional Rayleigh-Stokes problem (Mohebbi *et al.* 2013), and the heat conduction equation (Han & Dai 2013). To the best of our knowledge, no work has been reported until now on the higher order compact finite difference schemes for solving Burgers-Huxley equation.

In this paper, we present a new modified method for solving generalized Burgers-Huxley equation. The compact finite difference approach has been used in attempting to numerically solve the Burgers-Huxley equation. The numerical results have been compared with the exact solutions to verify the accuracy of the current method. In section 2 we introduce the generalized Burgers-Huxley equation. The derivation of the compact finite difference scheme for solving the generalized Burgers-Huxley equation was discussed in section 3. In section 4, we present the computational results and demonstrate the accuracy of the proposed scheme by presenting some numerical examples. We conclude our finding in section 5.

2. The Generalized Burgers-Huxley Equation

The generalized Burgers-Huxley equation with the initial and boundary conditions is as follows (Sari *et al.* 2011):

$$u_t + \alpha u^\delta u_x - u_{xx} - \beta u(1 - u^\delta)(u^\delta - \gamma) = 0, \quad 0 \leq x \leq 1, t \geq 0 \quad (1)$$

with the initial condition

$$u(x, 0) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} \tanh(a_1 x) \right)^{1/\delta} \quad (2)$$

and the boundary conditions

$$u(0,t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} \tanh(-a_1 a_2 t) \right)^{1/\delta} \quad (3)$$

and

$$u(1,t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} \tanh(a_1 (1 - a_2 t)) \right)^{1/\delta} \quad (4)$$

The exact solution of Eq. (1) is

$$u(x,t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} \tanh(a_1 (x - a_2 t)) \right)^{1/\delta} \quad (5)$$

where

$$\begin{aligned} a_1 &= \frac{-\alpha\delta + \delta\sqrt{\alpha^2 + 4\beta(1+\delta)}}{4(1+\delta)}\gamma, \\ a_2 &= \frac{\gamma a}{1+\delta} - \frac{(1+\delta-\gamma)(-\alpha + \sqrt{\alpha^2 + 4\beta(1+\delta)})}{2(1+\delta)} \end{aligned} \quad (6)$$

where α, β, γ and δ are parameters that $\beta \geq 0$, $\delta > 0$, $u(x,t)$ is the dependent variable, x is the spatial coordinate and t is the time variable.

Eq. (1) describes the nonlinear wave processes in physics, economics and ecology. When $\alpha = 0, \delta = 1$, it is the Huxley equation, which describes nerve pulse propagation in nerve fibres and wall motion in liquid crystals. If $\beta = 0, \delta = 1$, Eq. (1) is reduced to the model describing the far field of wave propagation in nonlinear dissipative systems. As in Sari *et. al.* (2011), the Eq. (1) is reduced to the Fitzhugh-Nagoma equation when $\alpha = 0$.

3. A Compact Finite Difference Scheme

According to Spitz (1995), the basic approach of higher-order compact difference schemes is to introduce the compact finite difference approximations to the differential equations and then by repeated differentiation and associated compact differencing, a new high-order compact scheme will be developed that incorporates the effect of the leading truncation error terms in the standard method.

By manipulating the Taylor series expansion in two independent variables (Mohebbi *et. al.* 2013), it can be found that:

$$u(x+h,t) = u(x,t) + \frac{h}{1!}u_x + \frac{h^2}{2!}u_{xx} + \frac{h^3}{3!}u_{xxx} + O(h^4) \quad (7)$$

Known, the first order forward difference (u_x),

$$u_x = \frac{u(x+h,t) - u(x,t)}{h} - \frac{h}{2}u_{xx} + O(h^2) \quad (8)$$

Since the basic idea of compact approach is to find the compact approximants to the derivatives term in the truncation error by utilizing the governing equation, we isolate u_{xx} in Eq. (1),

$$u_{xx} = u_t + \alpha u^\delta u_x - \beta u(1-u^\delta)(u^\delta - \gamma) \quad (9)$$

and substitute into Eq. (8) gives,

$$u_x = \frac{u(x+h,t) - u(x,t)}{h} - \frac{h}{2}(u_t + \alpha u^\delta u_x - \beta u(1-u^\delta)(u^\delta - \gamma)) \quad (10)$$

Then, we substitute Eq. (10) into Eq. (1),

$$\begin{aligned} & u_t + \alpha u^\delta \left(\frac{u(x+h,t) - u(x,t)}{h} - \frac{h}{2}(u_t + \alpha u^\delta u_x - \beta u(1-u^\delta)(u^\delta - \gamma)) \right) \\ & - u_{xx} - \beta u(1-u^\delta)(u^\delta - \gamma) = 0 \end{aligned} \quad (11)$$

The classical finite difference method was employed to discretize the partial differential u_t, u_x, u_{xx} in Eq. (11).

$$\begin{aligned} & \frac{u(x,t+h) - u(x,t)}{h} \\ & + \alpha u^\delta \left(\frac{u(x+h,t) - u(x,t)}{h} - \frac{h}{2}(u_t + \alpha u^\delta u_x - \beta u(1-u^\delta)(u^\delta - \gamma)) \right) \\ & - \frac{u(x+2h,t) - 2u(x+h,t) + u(x,t)}{h^2} - \beta u(1-u^\delta)(u^\delta - \gamma) = 0 \end{aligned} \quad (12)$$

Simplify and isolate $u(x+2h,t)$ in Eq. (12) gives,

$$\begin{aligned} u(x+2h,t) &= 2u(x+h,t) - u(x,t) \\ &+ h(u(x,t+h) - u(x,t) + \alpha u^\delta u(x+h,t) - \alpha u^\delta u(x,t)) \\ &- \frac{1}{2}h^2 \alpha u^\delta \left(-\alpha u^\delta u(x,t) - h\beta u(1-u^\delta)(u^\delta - \gamma) \right) \\ &- h^2 \beta u(1-u^\delta)(u^\delta - \gamma) \end{aligned} \quad (13)$$

We rewrite Eq. (13) in index form,

$$\begin{aligned}
 & U_{i+2,j} \\
 &= 2U_{i+1,j} - U_{i,j} + h(U_{i,j+1} - U_{i,j} + \alpha U_{i,j}^\delta U_{i+1,j} - \alpha U_{i,j}^\delta U_{i,j}) \\
 & - \frac{1}{2}h^2\alpha U_{i,j}^\delta (U_{i,j+1} - U_{i,j} + \alpha U_{i,j}^\delta U_{i+1,j} - \alpha U_{i,j}^\delta U_{i,j} - h\beta U_{i,j}(1 - U_{i,j}^\delta)(U_{i,j}^\delta - \gamma)) \\
 & - h^2\beta U_{i,j}(1 - U_{i,j}^\delta)(U_{i,j}^\delta - \gamma)
 \end{aligned} \tag{14}$$

The new modified scheme is an implicit scheme.

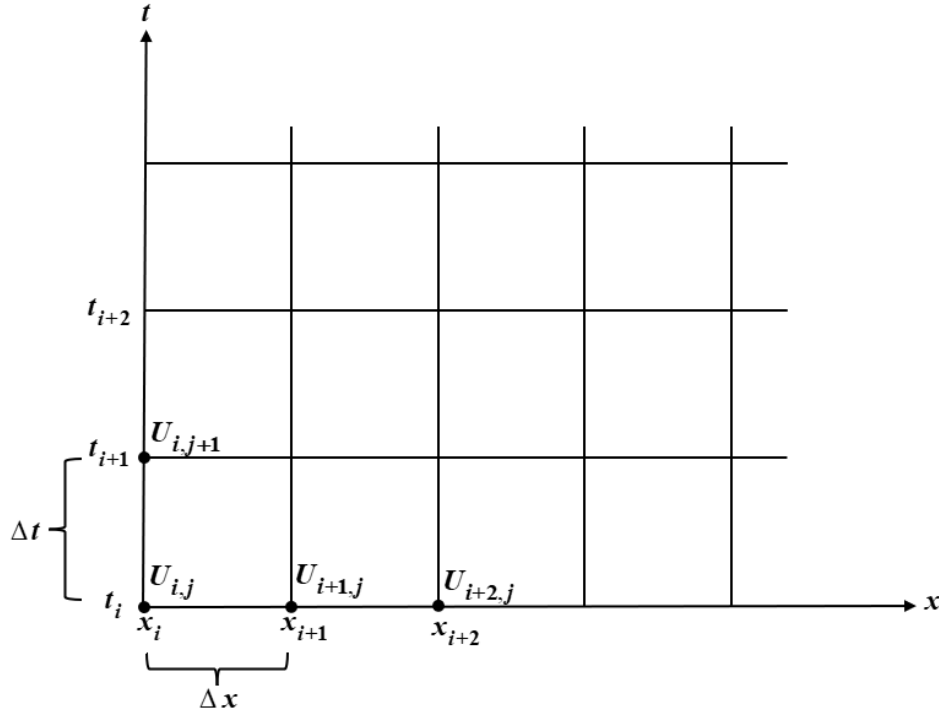


Figure 1: Finite grid network based on the Scheme (14)

4. Numerical Experiments

By the assistance of MATLAB R2010a software, we will compute all the numerical solutions of Scheme (14) with different value of the step size, h .

Example 1. The numerical experiments are conducted for the generalized Burgers-Huxley Eq. (1) for various values of x and t with $\delta=1$, the advection coefficient, $\alpha=1$, the coefficient of reaction, $\beta=1$ and $\gamma=0.001$. For the grid sizes, $h = 0.001, 0.0025, 0.025$ and 0.05 , we obtained the approximate solutions of Scheme (14) for Example 1 (Table 1).

Table 1: Approximate solution at various grid points for Example 1 with $h = 0.001, 0.0025, 0.025$ and 0.05 .

Grid Points		$u(0.25,0.25)$	$u(0.50,0.50)$	$u(0.75,0.75)$	$u(1.00,1.00)$
Exact Solution		5.0006247e-04	5.0012494e-04	5.0018741e-04	5.0024988e-04
Scheme (14)	$h = 0.0010$	5.0006286e-04	5.0012579e-04	5.0018770e-04	5.0024964e-04
	$h = 0.0025$	5.0006311e-04	5.0012689e-04	5.0018937e-04	5.0025016e-04
	$h = 0.0250$	5.0007181e-04	5.0014630e-04	5.0019688e-04	5.0024389e-04
	$h = 0.0500$	5.0007160e-04	5.0015775e-04	5.0023284e-04	5.0027801e-04

The approximate solutions show very good agreement with the exact solutions. It can be seen that as the grid size h decreased, the approximation generally becomes closer to the exact. We can conclude that the scheme is accurate.

Table 2: Comparison of the average relative errors of Example 1 with various values of h .

h	CFD scheme	Standard Scheme
0.005	3.802653867e-06	3.802654495e-06
0.010	8.950869961e-06	8.950883902e-06
0.050	5.593685183e-05	5.593688983e-05
0.100	1.006133346e-04	1.006150310e-04
0.250	1.427023933e-04	1.427110860e-04

Comparative study between the current scheme and standard scheme (Table 2) indicates that the CFD scheme produce more accurate solution.

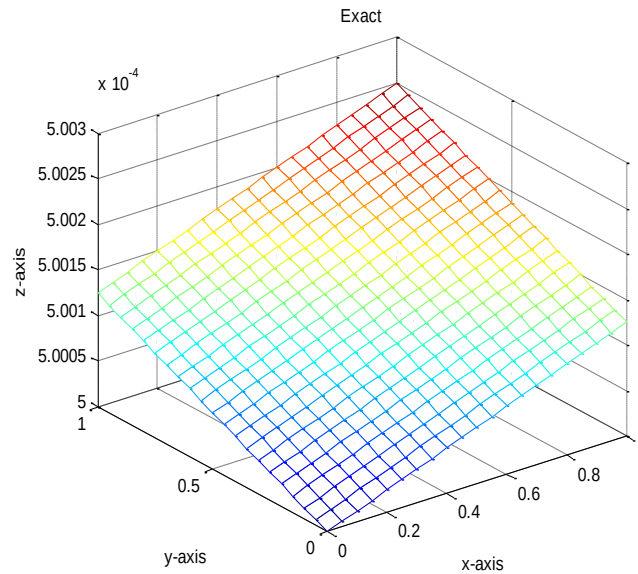
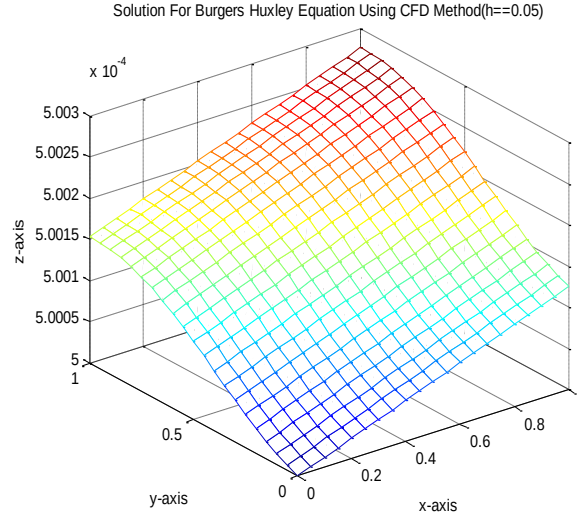


Figure 2: Exact solution of Example 1


Figure 3: Solution of Example 1 using CFD scheme for $h = 0.05$

From comparison of the graph (Figure 2 and Figure 3), it shows that the approximation close to the exact solution and it's free of obvious artificial oscillation.

Example 2. We consider Eq. (1) with the initial and boundary conditions (2) – (4) and exact solution (5) – (6) with $\alpha = \beta = 1$, $\delta = 2$, and $\gamma = 0.01$.

The numerical experiments are conducted at selected grid points for different grid size, h . The approximate solutions for various values of the grid size, h are presented in Table 3. We can conclude that the approximations converge to the exact as h is decreased.

The current results are seen to be more accurate than other results. Besides, Table 4 shows that the average relative error becomes smaller as the mesh size is decreased. Hence, the proposed scheme is accurate and converge.

Table 3: Approximate solution at various grid points for Example 2 with $h = 0.001, 0.0025, 0.025$ and 0.05 .

Grid Points		$u(0.25,0.25)$	$u(0.50,0.50)$	$u(0.75,0.75)$	$u(1.00,1.00)$
Exact Solution		7.07987169e-02	7.08866451e-02	7.09744623e-02	7.10621677e-02
Scheme (14)	$h = 0.0010$	7.07993258e-02	7.08880090e-02	7.09751395e-02	7.10622624e-02
	$h = 0.0025$	7.07996844e-02	7.08896231e-02	7.09775894e-02	7.10630166e-02
	$h = 0.0250$	7.08124873e-02	7.09181645e-02	7.09886003e-02	7.10537437e-02
	$h = 0.0500$	7.08121811e-02	7.09349980e-02	7.10414443e-02	7.11038425e-02

Table 4: Comparison of the average relative errors of Example 2 with various values of h .

h	CFD scheme	Standard Scheme
0.005	4.010821069e-05	4.010825473e-05
0.010	9.383470496e-05	9.383607688e-05
0.050	5.815621652e-04	5.815660095e-04
0.100	1.044838710e-03	1.045015202e-03
0.250	1.480443944e-03	1.481347908e-03

Furthermore, the approximate solution and the actual solution of the generalized Burgers-Huxley equation were illustrated in the graphs.

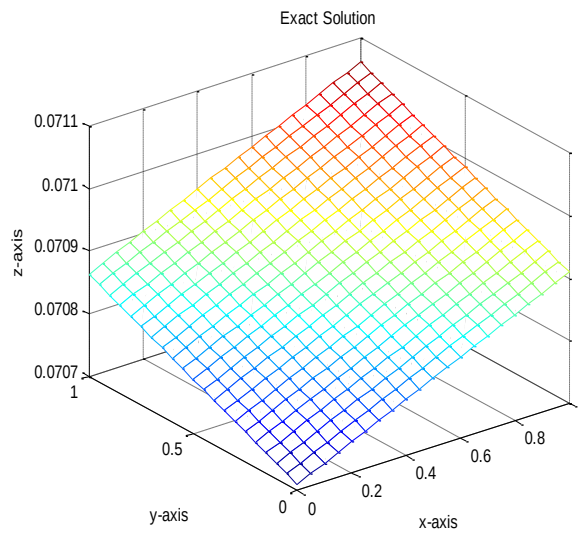


Figure 4: Exact solution of Example 2

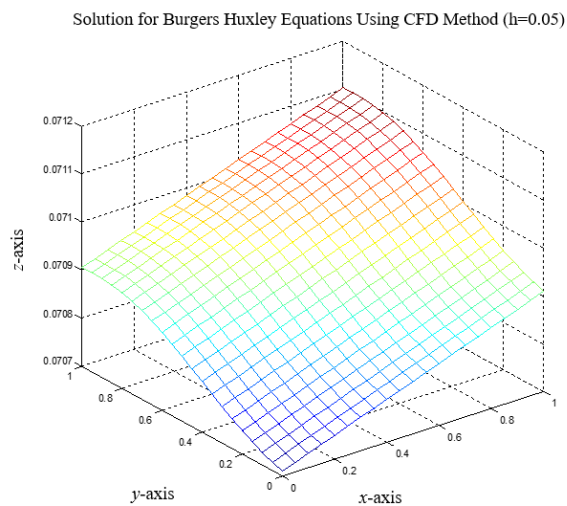


Figure 5: Solution of Example 2 using CFD scheme for $h = 0.05$

The graphs of the approximation (Figure 5) are compared with the graph of exact solution (Figure 4). The comparison shows that the difference between approximate solution and the exact solution are too small. Additionally, the solutions do not oscillate and converge.

Example 3. We consider the Burgers-Huxley, Eq. (1) with the given initial and boundary conditions (2) – (4) with $\alpha = \beta = 1$, $\delta = 4$ and $\gamma = 0.01$.

Table 5 shows the approximate solution of Example 3 with various step sizes. The approximation seemed closer to the exact solution. The average relative errors of the CFD scheme are seen to be more accurate compared to the relative errors of the standard scheme. As the grid sizes tend to zero, the average relative errors become smaller. Hence, the accuracy of the solution increased as the grid size is decreased.

Table 5: Approximate solution for various values of x and t with $\alpha = \beta = 1, \delta = 4$ and $\gamma = 0.01$ for $h = 0.001, 0.0025, 0.025$ and 0.05 .

Grid Points		$u(0.25,0.25)$	$u(0.50,0.50)$	$u(0.75,0.75)$	$u(1.00,1.00)$
Exact Solution		2.66245902e-01	2.66575762e-01	2.66904369e-01	2.67231718e-01
Scheme (14)	$h = 0.0010$	2.66248481e-01	2.66581847e-01	2.66908651e-01	2.67234982e-01
	$h = 0.0025$	2.66249891e-01	2.66588193e-01	2.66918271e-01	2.67237935e-01
	$h = 0.0250$	2.66300346e-01	2.66700419e-01	2.66961355e-01	2.67201456e-01
	$h = 0.0500$	2.66299136e-01	2.66766639e-01	2.67168780e-01	2.67397460e-01

Table 6: Comparison of the average relative errors of Example 3 with various values of h .

h	CFD scheme	Standard scheme
0.005	4.333275313e-05	4.333281105e-05
0.010	9.993407697e-05	9.993552476e-05
0.050	6.107713113e-04	6.107751573e-04
0.100	1.095538472e-03	1.095723418e-03
0.250	1.552148112e-03	1.553096631e-03

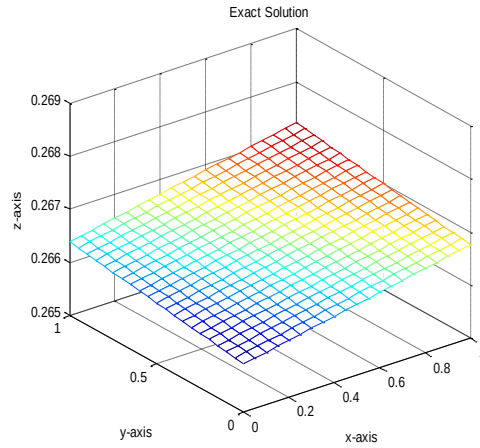


Figure 6: Exact solution of Example 3

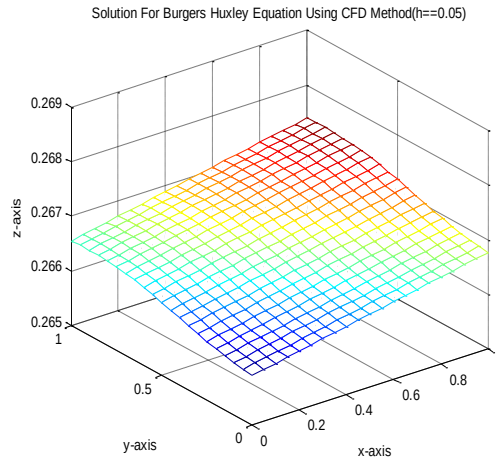


Figure 7: Solution of Example 3 using CFD scheme for $h = 0.05$

Comparative study between Figure 6 and Figure 7 shows that the error between these two solutions is significantly small, it's does reflect the accuracy of Scheme (14).

5. Conclusions

In this paper, the compact finite difference scheme has been proposed to solve the generalized Burgers-Huxley equation. The numerical results showed a very good agreement between the approximate and the exact solution. The accuracy of the solution was improves as the mesh size, h is reduced. Therefore, the compact finite difference technique is trusted for solving the generalized Burgers-Huxley equation. The true value are needed as initial start at the second column of finite grid network, Figure 1.

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