

DIRAC DELTA AND GAUSSIAN DISTRIBUTION FOR SOLVING THE THREE-DIMENSIONAL TRANSIENT GROUNDWATER FLOW EQUATION

(Taburan Delta Dirac dan Gaussian untuk Menyelesaikan Persamaan Aliran Air Bawah
Tanah Tiga Dimensi)

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ABSTRACT

This paper presents an analytical solution to the three-dimensional transient groundwater flow equation in a homogeneous, isotropic, and infinite confined aquifer using the Fourier transform. The governing equation is solved based on two initial conditions: a Dirac Delta distribution representing an instantaneous point source, and a Gaussian distribution representing a smooth, localized initial perturbation. To verify the accuracy of the analytical solution of the Gaussian initial condition, a numerical solution using the Finite Difference Method with a Forward Time Centred Space scheme is performed with comparisons made at time various times. Errors are quantified using the L^2 norm. A grid convergence study confirms second-order spatial convergence which is consistent with the theoretical truncation error of the central difference scheme. The analytical solutions provide exact deterministic benchmarks for three-dimensional transient groundwater flow problems establishing a foundation for future stochastic extensions.

Keywords: groundwater flow; Dirac Delta distribution; Gaussian distribution

ABSTRAK

Kertas ini membentangkan penyelesaian analitik bagi persamaan aliran air bawah tanah tiga dimensi dalam akuifer terkurung yang homogen, isotropik dan tak terhingga dengan menggunakan jelmaan Fourier. Persamaan menakluk diselesaikan bagi dua syarat awal, iaitu taburan Delta Dirac yang mewakili sumber titik serta-merta, dan taburan Gaussian yang mewakili usikan awal yang licin dan setempat. Bagi mengesahkan ketepatan penyelesaian analitik untuk syarat awal Gaussian, penyelesaian berangka yang diperolehi melalui kaedah beza terhingga dengan skema masa ke hadapan ruang berpusat dibandingkan dengan penyelesaian analitik pada pelbagai masa. Ralat diukur menggunakan norma L^2 . Kajian penumpuan grid mengesahkan penumpuan ruang peringkat kedua, yang selaras dengan ralat pemenggalan teori bagi skema beza berpusat. Penyelesaian analitik ini menyediakan penanda aras berketentuan yang tepat bagi masalah aliran air bawah tanah tiga dimensi, sekali gus membentuk asas bagi lanjutan stokastik pada masa hadapan.

Kata kunci: aliran air bawah tanah; taburan Delta Dirac; taburan Gaussian

1. Introduction

Groundwater flow within aquifers is governed by complex physical processes and interactions that come from the heterogeneity of geological medium, variability in hydraulic parameters, and dynamic boundary conditions (Medici *et al.* 2019). In many regions, including Malaysia, aquifer systems are not perfectly uniform or free from external influences such as pumping, localized recharge, or temporal fluctuations (Wu *et al.* 2021). The three-dimensional governing equations for groundwater flow must, therefore, accommodate a range of complexities, which

include spatially varying hydraulic conductivity, transient or time-dependent boundary conditions, and sources or sinks that represent wells or infiltration scenarios. These complexities combine to produce a broad variety of problem formulations and each of these problems requires a specific analytical or numerical solution (Feng *et al.* 2020).

The general three-dimensional transient groundwater flow equation provides a fundamental framework for describing subsurface flow processes. It is derived from the principles of mass conservation and Darcy's law. In simplified cases where assumptions such as steady-state condition, constant hydraulic conductivity, uniform specific storage, and zero external sources or sink are valid, analytical solutions can be derived. Early analytical solutions in groundwater hydrology established foundational concepts of steady and transient radial flow to wells with later work introducing time dependent drawdowns through the storage term.

These models, which were later extended by various authors, typically assume homogeneity of the aquifer and isotropic hydraulic conductivity (Bear & Cheng 2010). Such analytical models have also been applied to aquifers with well-defined lateral boundaries, to idealised recharge conditions and to periodic forcing such as tidal fluctuations. Sedghi and Zhan (2025), for instance, derived a three-dimensional solution for flow to a partially penetrating well in a wedge-shaped alluvial fan aquifer bounded by two intersecting faults, obtained by means of Laplace and Fourier transforms, while Cheng *et al.* (2025) derived an analytical solution for groundwater dynamics under periodic harmonic forcing. Moreover, analytical solutions provide a framework for validating numerical models and for systematically studying the effects of additional complexities, such as heterogeneity or transient forcing, by relaxing some of the simplifying assumptions (Wang & Luo 2021). Solutions based on the Fourier transform have been applied to linear flow equations, and Fourier transform techniques in particular allow such problems to be solved by decomposing complex boundary conditions into their harmonic components. Their application to the three-dimensional transient groundwater flow equation, however, remains limited. Despite their advantages, analytical approaches are often constrained by simplifying assumptions, such as homogeneity or idealised aquifer geometries (Wen *et al.* 2024). Nevertheless, when these conditions are met, or can be reasonably approximated, the analytical solutions obtained serve as an alternative to numerical simulations.

This paper focuses on deriving analytical solutions to the three-dimensional transient groundwater flow equation using Fourier transform. This work extends the framework presented by Tawfiqa and Jabber (2017) by applying the methodology to the full three-dimensional governing equation. The novelty of the current work lies in extending the previous methodology by explicitly formulating a physical model of the aquifer system, and in the analysis of the solution under two fundamental types of initial condition: the Dirac Delta distribution and the Gaussian distribution. The Dirac Delta distribution represents the response to an instantaneous point perturbation while the Gaussian distribution represents a smooth, localized, initial perturbation of the hydraulic head which correspond to a brief infiltration event. These solutions serve as a tool for solving problems with arbitrary hydraulic parameters. To verify the accuracy of the Gaussian initial conditions, the analytical solution is compared against a numerical scheme using the Finite Difference Method, consistent with established methodology (Seyedpour *et al.* 2021; Tracy & Vahedifard 2023) and the error metric is quantified using the L^2 norm error.

In recent decades, research in hydrology has largely focused on stochastic modelling and numerical simulations especially when dealing with uncertain aquifer properties or highly heterogeneous systems (Quilty *et al.* 2022; Reichert *et al.* 2021; Shen *et al.* 2022). However, the development of analytical and semi-analytical techniques remains important for obtaining benchmark solutions and improving the understanding of complex mathematical models. Such

techniques have also been developed for other classes of differential and integro-differential problems, demonstrating their usefulness in obtaining approximate analytical solutions to complex governing equations (Eshkuvatov *et al.* 2025). This work explores the application of Fourier transforms to derive an analytical solution for the three-dimensional transient groundwater flow equation. The aim is to advance analytical groundwater modelling by: (1) deriving Fourier-based solutions for the Dirac Delta distribution and the Gaussian distribution, (2) validating the analytical solution against a numerical scheme, the finite difference method, and (3) quantifying the errors using the L^2 norm error. The results serve as a benchmark for systems with localized or transient forcing.

2. Mathematical Model

2.1. Darcy's Law and the Continuity Equation

Groundwater flow through porous media is governed by a set of Partial Differential Equations (PDEs) based on Darcy's Law and the continuity equation. Darcy's Law, formulated by Henry Darcy in 1856, describes the flow of fluid through a porous medium by linearly relating the volume flow rate of the fluid to the hydraulic head difference via the hydraulic conductivity. Mathematically, Darcy's Law is expressed as,

$$q = -k\nabla h, \tag{1}$$

where

q is the specific discharge or Darcy's velocity (m/s),

k is the hydraulic conductivity (m/s),

h is the hydraulic head (m), and

∇h represents the gradient of the hydraulic head.

This relationship indicates that groundwater flows from regions of high hydraulic head to regions of low hydraulic head, with the hydraulic conductivity quantifying the ease of water movement through a given medium. The hydraulic conductivity depends on the permeability of the material and fluid properties such as viscosity and density.

The conservation of mass is expressed through the continuity equation, which ensures that the water entering a control volume equals the water leaving it, plus any changes in storage. By combining Darcy's Law and the continuity equation, the governing equation for groundwater flow can be derived.

2.2. Problem Setup

A mathematical model is developed to describe the hydraulic head distribution in a homogeneous, confined aquifer with an infinite domain, as illustrated in Figure 1. The assumption follows the pioneering work by Neuman and Witherspoon (1969).

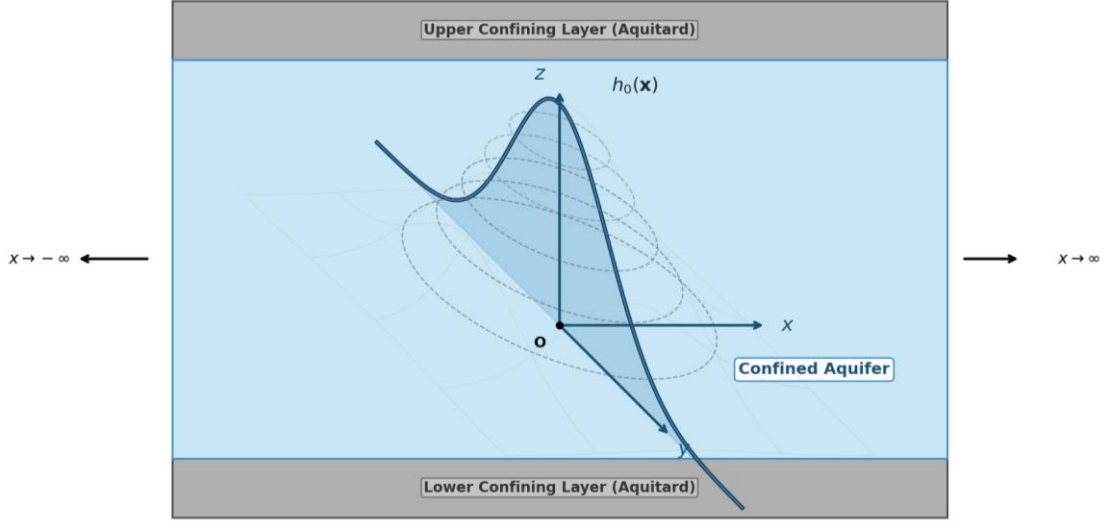


Figure 1: Schematic of the confined aquifer system with infinite lateral extent. The initial hydraulic head perturbation $h_0(\mathbf{x})$ is applied at the origin within the aquifer, bounded by upper and lower confining layers (aquitards)

The confined aquifer is approximated as infinite since boundary effects are negligible over the studied timescale. Thus, we assume an infinite domain on $(x, y, z \in \mathbb{R}^3)$ which simplifies the analysis of large-scale aquifers where boundary effects are negligible. Physically, this means that the aquifer is large enough that the boundaries are irrelevant over practical timescales (Alecsa *et al.* 2020; Konikow & Neuzil 2007).

The governing differential equation, which describes the three-dimensional transient groundwater flow equation in terms of head on a homogeneous and isotropic aquifer, is as follows,

$$k \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \right) = S_s \frac{\partial h}{\partial t}, \quad (2)$$

where

k is the hydraulic conductivity,

h is the hydraulic head,

t is the time, and

S_s is the specific storage.

The equation describes flow under confined conditions where the aquifer is fully saturated and bounded above and below by impermeable confining layers. The hydraulic head $h(x, y, z, t)$ represents the piezometric head. Thus, the initial condition is set to h_0 at $t = 0$, and expressed as,

$$h(x, y, z, t = 0) = h_0(x, y, z), \quad (3)$$

where h_0 is sufficiently smooth and integrable.

With an infinite domain, boundary conditions are replaced by behaviour conditions at infinity,

$$\lim_{|x|, |y|, |z| \rightarrow \infty} h(x, y, z, t) = h_\infty, \quad (4)$$

or more generally, h and its derivatives vanish at infinity. The Dirichlet boundary conditions will be specified as follows,

$$h(\pm\infty, y, z, t) = 0, \quad (5)$$

$$h(x, \pm\infty, z, t) = 0, \quad (6)$$

$$h(x, y, \pm\infty, t) = 0. \quad (7)$$

There are two merits of using an infinite domain. Firstly, the absence of physical boundaries eliminates complex boundary interactions that arise from rivers or impermeable barriers. This simplifies the governing equation. Secondly, infinite domains often yield exact closed-form solutions. This is generally not obtainable for finite domains with arbitrary boundary geometries. Additionally, even when the physical aquifer is finite, the infinite-domain solution provides a valid approximation at early times before the effects of distant boundaries propagate to the region of interest.

The problem posed above will be solved using the Fourier transform.

3. Analytical Solution using Fourier Transform

This section details the core contribution of this paper: the derivation of the analytical solution for three-dimensional transient groundwater flow using the Fourier transform.

3.1. Principles of the Fourier Transform

The Fourier transform is a mathematical tool that decomposes a function of space or time into its constituent frequencies, thereby mapping it from the spatial or temporal domain to the frequency domain. In the context of solving PDEs like the groundwater flow equation, the Fourier transform simplifies spatial derivatives into algebraic terms, making the equation less complex to solve (Chatzigeorgiou *et al.* 2022; Rapp 2017).

The Fourier transform of a function $f(x)$ is defined as,

$$F(a) = \int_{-\infty}^{\infty} f(x) \exp\{-iax\} dx, \quad (8)$$

where

$F(a)$ is the Fourier transform of $f(x)$,

a is the wavenumber, which is the frequency variable associated with x ,

$i = \sqrt{-1}$ is the imaginary unit, and

$\exp\{-iax\}$ is the complex exponential that captures the frequency content of $f(x)$.

The inverse Fourier transform is used to reconstruct $f(x)$ from $F(a)$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(a) \exp\{iax\} da. \quad (9)$$

Two key properties of the Fourier transform are, firstly, corresponding to the time derivative, where,

$$F\left(\frac{\partial h}{\partial t}\right) = \frac{\partial \hat{h}}{\partial t}, \quad (10)$$

and secondly, corresponding to the second spatial derivative, where

$$F\left(\frac{\partial^2 h}{\partial x^2}\right) = -a_x^2 \hat{h}(a_x, a_y, a_z, t). \quad (11)$$

For three-dimensional functions $h(x, y, z)$, the Fourier transform extends naturally to

$$\hat{h}(a_x, a_y, a_z, t) = \iiint_{\mathbb{R}^3} h(x, y, z, t) \exp\{-i(a_x x + a_y y + a_z z)\} dx dy dz, \quad (12)$$

where a_x, a_y, a_z are the wavenumbers corresponding to the x -, y - and z -directions.

3.2. Fourier Transformation of the Partial Differential Equation

The transient three-dimensional groundwater flow equation for a homogeneous and isotropic confined aquifer is given by,

$$S_s \frac{\partial h}{\partial t} = k \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \right). \quad (13)$$

To apply the Fourier transform, take the transform of both sides with respect to the spatial variables, x , y , and z . For the temporal variables, apply the first property of the Fourier transform as in Eq. (10),

$$F\left(S_s \frac{\partial h}{\partial t}\right) = S_s \frac{\partial \hat{h}}{\partial t}, \quad (14)$$

where $\hat{h}(a_x, a_y, a_z, t)$ is the Fourier transform of $h(x, y, z, t)$.

For the spatial derivatives, the second property of the Fourier transform, as in Eq. (11), gives the transform of the second derivative with respect to x . Similarly, for the y -direction

$$F\left(\frac{\partial^2 h}{\partial y^2}\right) = -a_y^2 \hat{h}(a_x, a_y, a_z, t), \quad (15)$$

and the z -direction,

$$F\left(\frac{\partial^2 h}{\partial z^2}\right) = -a_z^2 \hat{h}(a_x, a_y, a_z, t). \quad (16)$$

Combining these results, the transformed equation becomes

$$S_s \frac{\partial \hat{h}}{\partial t} = -k(a_x^2 + a_y^2 + a_z^2) \hat{h}(a_x, a_y, a_z, t). \quad (17)$$

Equation (17) is an Ordinary Differential Equation (ODE) in the frequency domain, which is rearranged as

$$\frac{\partial \hat{h}}{\partial t} = -\frac{k}{S_s} (a_x^2 + a_y^2 + a_z^2) \hat{h}(a_x, a_y, a_z, t). \quad (18)$$

Equation (18) is a separable ODE; the derivation steps are included in Appendix A, and its solution is

$$\hat{h}(a_x, a_y, a_z, t) = A \exp \left\{ -\frac{k}{S_s} (a_x^2 + a_y^2 + a_z^2) t \right\}, \quad (19)$$

where $A = e^C$. Using the transformed initial condition,

$$\hat{h}(a_x, a_y, a_z, t = 0) = \hat{h}_0(a_x, a_y, a_z). \quad (20)$$

The solution is an exponential decay of the form,

$$\hat{h}(a_x, a_y, a_z, t) = \hat{h}_0(a_x, a_y, a_z) \exp \left\{ -\frac{k}{S_s} (a_x^2 + a_y^2 + a_z^2) t \right\}, \quad (21)$$

where $\hat{h}_0(a_x, a_y, a_z)$ is the Fourier transform of the initial hydraulic head distribution $h_0(x, y, z)$. This solution describes how the hydraulic head evolves over time in the frequency domain.

The solution in Eq. (21) is expressed in the frequency domain. To return to the spatial domain, the inverse Fourier transform is applied,

$$h(x, y, z, t) = \frac{1}{(2\pi)^3} \iiint_{\mathbb{R}^3} \hat{h}(a_x, a_y, a_z, t) \exp\{i(a_x x + a_y y + a_z z)\} da_x da_y da_z. \quad (22)$$

Substituting the solution for $\hat{h}(a_x, a_y, a_z, t)$ from Eq. (21) into Eq. (22) gives,

$$h(x, y, z, t) = \frac{1}{(2\pi)^3} \iiint_{\mathbb{R}^3} \hat{h}_0(a_x, a_y, a_z) \exp \left\{ -\frac{k}{S_s} |\mathbf{a}|^2 t \right\} \exp\{i\mathbf{a} \cdot \mathbf{x}\} da_x da_y da_z, \quad (23)$$

where $|\mathbf{a}|^2 = a_x^2 + a_y^2 + a_z^2$ and $\mathbf{a} \cdot \mathbf{x} = a_x x + a_y y + a_z z$.

The integral in Eq. (23) represents the hydraulic head at any point in space and time, given an initial distribution of head values. The solution shows that the hydraulic head propagates and dissipates over time, influenced by the diffusivity term k/S_s . High hydraulic conductivity k or low specific storage S_s results in faster propagation and dissipation of hydraulic head perturbations.

3.3. Inverse Fourier Transform and the General Solution

To express the solution entirely in terms of the initial condition $h_0(x, y, z)$, we substitute its Fourier transform $\hat{h}_0(a_x, a_y, a_z)$ where $\hat{h}_0$ is computed using auxiliary variables x' , y' and z' . These dummy integration variables distinguish the initial condition's coordinates from the observation coordinates (x, y, z) at time t .

$$\hat{h}_0(a_x, a_y, a_z) = \iiint_{\mathbb{R}^3} h_0(x', y', z') \exp\{-i(a_x x' + a_y y' + a_z z')\} dx' dy' dz'. \quad (24)$$

Substitute Eq. (24) into Eq. (23) gives

$$h(x, y, z, t) = \frac{1}{(2\pi)^3} \iiint_{\mathbb{R}^3} \left[\iiint_{\mathbb{R}^3} h_0(x', y', z') \exp\{-i(a_x x' + a_y y' + a_z z')\} dx' dy' dz' \right] \exp \left\{ -\frac{k}{S_s} |\mathbf{a}|^2 t + i\mathbf{a} \cdot \mathbf{x} \right\} da_x da_y da_z. \quad (25)$$

Reorganising the integrals in Eq. (25) gives

$$h(x, y, z, t) = \iiint_{\mathbb{R}^3} h_0(x', y', z') \left[\frac{1}{(2\pi)^3} \iiint_{\mathbb{R}^3} \exp \left\{ -\frac{k}{S_s} (a_x^2 + a_y^2 + a_z^2) t + i(a_x(x - x') + a_y(y - y') + a_z(z - z')) \right\} da_x da_y da_z \right] dx' dy' dz'. \quad (26)$$

The inner bracket of Eq. (26) is the inverse Fourier transform of a Gaussian and in one dimension, the integral is of the form

$$\int_{-\infty}^{\infty} \exp \left\{ -\frac{k}{S_s} (a_x^2 t + i a_x (x - x')) \right\} da_x = \sqrt{\frac{\pi S_s}{kt}} \exp \left\{ \frac{-S_s (x - x')^2}{4kt} \right\}. \quad (27)$$

Extend Eq. (27) into three dimensions to obtain,

$$h(x, y, z, t) = \iiint_{\mathbb{R}^3} h_0(x', y', z') \frac{S_s}{(4\pi kt)^{3/2}} \exp \left\{ \frac{-S_s ((x - x')^2 + (y - y')^2 + (z - z')^2)}{4kt} \right\} dx' dy' dz'. \quad (28)$$

Let $\alpha = k/S_s$. Thus, for each h_0 , the general solution to the transient three-dimensional groundwater flow equation in Eq. (13) is,

$$h(x, y, z, t) = \iiint_{\mathbb{R}^3} h_0(x', y', z') G(x - x', y - y', z - z', t) dx' dy' dz', \quad (29)$$

where G is the Gaussian kernel

$$G(x, y, z, t) = \frac{1}{(4\pi \alpha t)^{3/2}} \exp \left\{ \frac{-(x^2 + y^2 + z^2)}{4\alpha t} \right\}. \quad (30)$$

3.4. Initial Condition Case Studies

In this section, the derived analytical solution is extended to two cases. The first being the initial head representing an instantaneous point source in the form of a Dirac Delta distribution, and the second case being the initial head representing a Gaussian distribution.

3.4.1. Case 1: Dirac Delta initial condition

Let the initial condition be,

$$h_0(x, y, z) = \delta(x)\delta(y)\delta(z), \quad (31)$$

where $\delta(x)$ is the one-dimensional Dirac Delta distribution, satisfying

$$\int_{-\infty}^{\infty} \delta(x) dx = 1 \text{ and } \delta(x) = 0 \text{ for } x \neq 0. \quad (32)$$

The product $\delta(x)\delta(y)\delta(z)$ defines a three-dimensional Dirac Delta distribution. In this case, the physical scenario represents an instantaneous point source at $x = 0, y = 0$, and $z = 0$.

0, such as the injection of water at initial head. The general solution is the convolution of h_0 with the Gaussian kernel G ,

$$h(x, y, z, t) = \iiint_{\mathbb{R}^3} \delta(x')\delta(y')\delta(z')G(x-x', y-y', z-z', t) dx' dy' dz'. \quad (33)$$

By the sifting property of the Dirac Delta, this will simplify the integral

$$h(x, y, z, t) = G(x-0, y-0, z-0, t) = G(x, y, z, t). \quad (34)$$

Thus, the hydraulic head governed by Eq. (13) with the Dirac Delta initial condition in Eq. (31) at time t is,

$$h(x, y, z, t) = \frac{1}{(4\pi\alpha t)^{3/2}} \exp \left\{ -\frac{x^2+y^2+z^2}{4\alpha t} \right\}, \quad (35)$$

where $\alpha = k/S_s$.

3.4.2. Case 2: Gaussian initial condition

Let the initial condition be

$$h_0(x, y, z) = A \exp \left\{ -\frac{(x^2+y^2+z^2)}{\sigma^2} \right\}, \quad (36)$$

where A is the amplitude and σ is the width of the Gaussian.

Since the initial condition at $t = 0$ is $h_0(x, y, z)$, then the solution follows from the general solution in Eq. (29) as the convolution of h_0 with the Gaussian kernel G in Eq. (30),

$$h(x, y, z, t) = \frac{1}{(4\pi\alpha t)^{3/2}} \iiint_{\mathbb{R}^3} h_0(x', y', z') \exp \left\{ -\frac{(x-x')^2 + (y-y')^2 + (z-z')^2}{4\alpha t} \right\} dx' dy' dz'. \quad (37)$$

Substituting Eq.(36) into Eq.(37) gives

$$h(x, y, z, t) = \frac{A}{(4\pi\alpha t)^{3/2}} \iiint_{\mathbb{R}^3} \exp \left\{ -\frac{(x'^2+y'^2+z'^2)}{\sigma^2} \right\} \exp \left\{ -\frac{(x-x')^2 + (y-y')^2 + (z-z')^2}{4\alpha t} \right\} dx' dy' dz'. \quad (38)$$

To evaluate the integral in Eq. (38), combine the exponents of like terms into one quadratic form and since the exponent in the three-dimensional integral separates a product of three identical one-dimensional integrals, each for x, y, z , respectively, as below

$$h(x, y, z, t) = \frac{A}{(4\pi\alpha t)^{3/2}} \iiint_{\mathbb{R}^3} \exp \left\{ -\frac{x'^2}{\sigma^2} - \frac{(x-x')^2}{4\alpha t} \right\} \exp \left\{ -\frac{y'^2}{\sigma^2} - \frac{(y-y')^2}{4\alpha t} \right\} \exp \left\{ -\frac{z'^2}{\sigma^2} - \frac{(z-z')^2}{4\alpha t} \right\} dx' dy' dz'. \quad (39)$$

We can work out the convolution of two Gaussian functions in one dimension. Let

$$I(x) = \int_{-\infty}^{\infty} \exp \left\{ -\frac{x'^2}{\sigma^2} - \frac{(x-x')^2}{4\alpha t} \right\} dx' = \int_{-\infty}^{\infty} \exp \left\{ -\left(\frac{x'^2}{\sigma^2} + \frac{x'^2 - 2xx' + x^2}{4\alpha t} \right) \right\} dx'. \quad (40)$$

The evaluation of the Gaussian Convolution Integral is provided in Appendix B. We now have a standard Gaussian integral of the form

$$\int_{-\infty}^{\infty} \exp\{-pu^2\} du = \sqrt{\frac{\pi}{p}} \text{ for } p > 0. \quad (41)$$

Evaluating Eq. (40), therefore, yields the one-dimensional result

$$I(x) = \exp\left\{-\frac{x^2}{4\alpha t + \sigma^2}\right\} \sqrt{\pi \frac{4\alpha t \sigma^2}{4\alpha t + \sigma^2}}. \quad (42)$$

Since the convolution separates in three dimensions, substituting Eq. (42) into Eq. (39) gives the full solution for the Gaussian initial condition

$$h(x, y, z, t) = \frac{A}{(4\pi\alpha t)^{3/2}} \exp\left\{\frac{-x^2 - y^2 - z^2}{4\alpha t + \sigma^2}\right\} \left(\pi \frac{4\alpha t \sigma^2}{4\alpha t + \sigma^2}\right)^{\frac{3}{2}}. \quad (43)$$

Simplifying Eq. (43) gives the hydraulic head governed by Eq. (13) with the Gaussian initial condition in Eq. (36) at time t as

$$h(x, y, z, t) = A \left(\frac{\sigma^2}{\sigma^2 + 4\alpha t}\right)^{\frac{3}{2}} \exp\left\{-\frac{(x^2 + y^2 + z^2)}{\sigma^2 + 4\alpha t}\right\}. \quad (44)$$

4. Numerical Method

4.1. Finite Difference Discretization

The Finite Difference Method (FDM) is used as the numerical illustration to verify the analytical solutions obtained following established approaches in groundwater flow modelling (Seyedpour *et al.* 2021; Tracy & Vahedifard 2023). This method approximates derivatives in governing PDE using finite differences, which transforms the PDE into a system of algebraic equations to be solved numerically. The numerical approach is done for the Gaussian initial condition.

To validate the derived analytical solutions, the three-dimensional transient groundwater flow equations are solved numerically using the FDM with a Forward Time Centred Space (FTCS) scheme. The infinite domain boundary conditions, $h \rightarrow 0$ as $x, y, z \rightarrow \infty$, were approximated by a large, finite, computational domain.

The spatial domain for the numerical model was defined as $x \in [-L, L]$, $y \in [-L, L]$, and $z \in [-L, L]$, utilizing grid points $N = 101$ in each dimension. This results in a uniform spatial step size given by,

$$\Delta x = \Delta y = \Delta z = \frac{2L}{N-1}. \quad (45)$$

For temporal discretization, a forward Euler scheme was applied to the time derivative,

$$\frac{\partial h}{\partial t} \approx \frac{h^{n+1} - h^n}{\Delta t}, \quad (46)$$

and the time step Δt is chosen to satisfy the stability condition for the forward Euler scheme applied to the diffusion equation. As pointed out in Saad and Karam (2022), it is a necessary condition of convergence of the FDM when solving diffusion equations, and the time step is therefore adapted to the stability condition (Ranocha *et al.* 2022),

$$\Delta t \leq \frac{\Delta x^2}{6\alpha}, \quad (47)$$

where $\alpha = k/S_s$ is the hydraulic diffusivity.

The spatial derivatives are approximated using the central difference scheme,

$$\frac{\partial^2 h}{\partial x^2} = \frac{h_{i+1,j,k}^n - 2h_{i,j,k}^n + h_{i-1,j,k}^n}{\Delta x^2}, \quad (48)$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{h_{i,j+1,k}^n - 2h_{i,j,k}^n + h_{i,j-1,k}^n}{\Delta y^2}, \quad (49)$$

$$\frac{\partial^2 h}{\partial z^2} = \frac{h_{i,j,k+1}^n - 2h_{i,j,k}^n + h_{i,j,k-1}^n}{\Delta z^2}. \quad (50)$$

4.2. Error Analysis

Following the approach of Hsieh and Lee (2021), the discrepancy between the numerical and analytical solutions is quantified using the L^2 norm defined as

$$\|h_{num} - h_{anal}\|_{L^2} = \left(\iiint_{\mathbb{R}^3} |h_{num} - h_{anal}|^2 dx dy dz \right)^{\frac{1}{2}}. \quad (51)$$

This norm provides a measure of the overall accuracy of the numerical solution across the entire computational domain and indicates the average magnitude of the discrepancy between the two solutions.

5. Results and Discussion

This section presents the findings of the numerical validation of the analytical solutions. The three-dimensional FDM is implemented using the FTCS scheme to verify the analytical solutions derived. Additionally, a grid convergence study is conducted to assess the spatial accuracy and convergence rate of the numerical scheme.

5.1. General Setup and Parameters

The spatial domain is truncated to a finite interval of $L = 40$ m in each dimension for the numerical computations. The width is chosen in order to approximate an infinite domain. The parameters common to both case studies are a hydraulic conductivity $k = 10^{-4}$ m/s and a specific storage $S_s = 10^{-4}$ m^{-1} . This yields a diffusivity value of $\alpha = k/S_s = 1.0$ m^2/s . The numerical grid uses $N = 101$ points per dimension, which gives a uniform grid spacing of $\Delta x = \Delta y = \Delta z = 0.80$ m. An odd number of grid points was chosen to ensure that the origin coincides exactly with a grid node for accurate evaluation of peak values. The initial width $\sigma = 1$ m keeps the perturbation strongly localized relative to the domain, with $\sigma/L = 0.025$, while remaining adequately resolved.

5.2. Gaussian Initial Condition

For this case, the initial condition is a Gaussian head distribution given in Eq. (36). This initial condition physically represents a smooth, localized, Gaussian perturbation of the initial hydraulic head. Setting $A = 1$ m and $\sigma = 1$ m in the general Gaussian solution, Eq. (44), gives the analytical solution used for this case,

$$h(x, y, z, t) = \left(\frac{\sigma^2}{\sigma^2 + 4\alpha t} \right)^{\frac{3}{2}} \exp \left\{ -\frac{(x^2 + y^2 + z^2)}{\sigma^2 + 4\alpha t} \right\}. \quad (52)$$

The numerical and analytical solutions are compared $t = 0.1$ s, 1.0 s, and 10.0 s using the same hydraulic parameters. Table 1 presents the peak head values and the L^2 errors computed between the analytical and numerical results.

Table 1: Comparison of analytical and numerical results for the Gaussian initial condition

Time (s)	Analytical Peak	Numerical Peak	L^2 Error
0.1	0.603682	0.625023	0.029241
1.0	0.089443	0.086671	0.008646
10.0	0.003809	0.003788	0.000288

The numerical solution shows good agreement with the analytical solution at all three evaluation times. At $t = 0.1$ s, the numerical peak slightly overestimates the analytical value by approximately 3.5%. This overprediction at early times is a known characteristic of explicit numerical schemes applied to diffusion problems with steep gradients (Castiglioni *et al.* 2019).

At $t = 1.0$ s and $t = 10.0$ s, the agreement improves further. The numerical peaks underestimate the analytical value by 3.1% and 0.6%, respectively, and the L^2 error falls to 8.6×10^{-3} and 2.9×10^{-4} respectively. One advantage of the analytical framework lies in computational efficiency. The analytical solution requires only the direct evaluation of a closed-form expression at each grid point, which is computed within seconds for the full three-dimensional domain. The FDM simulation at $N = 101$ requires iterative time stepping over thousands of steps, with computational times increasing at higher times, reaching $t = 10.0$ s. The advantage becomes more pronounced for longer simulation times and finer grids.

Figures 2 to 4 illustrate this behavior. Figure 2 shows the analytical hydraulic head surface obtained from Eq. (52) at the three evaluation times, and the solution retains its Gaussian form while the peak decays and the perturbation spreads laterally. Figure 3 compares the analytical and numerical head profiles along the x -axis. The two curves are visually indistinguishable at $t = 1.0$ s and $t = 10.0$ s, and differ only near the peak at $t = 0.1$ s, which is consistent with the L^2 errors reported in Table 1. Figure 4 presents the same cross sections with the analytical solution and the numerical solution plotted separately and confirms that the solutions broaden and flatten at the same rate.

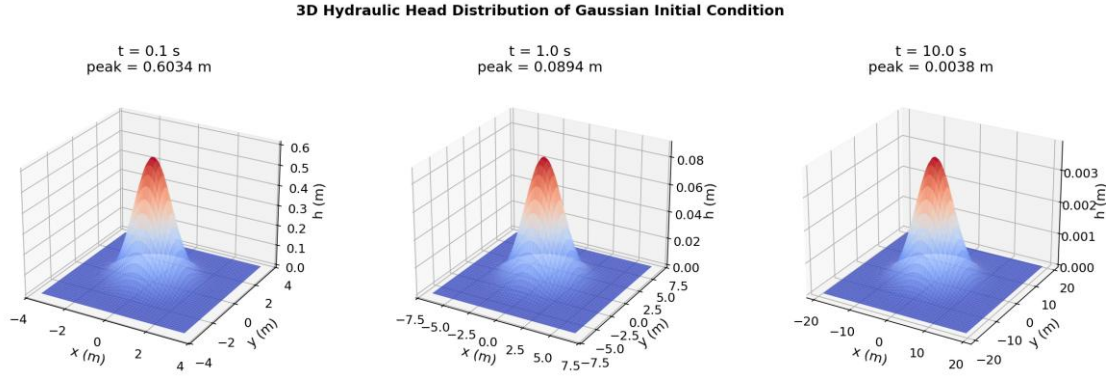


Figure 2: 3D analytical hydraulic head surface at $t = 0.1 \text{ s}$, 1.0 s , and 10.0 s for the Gaussian initial condition

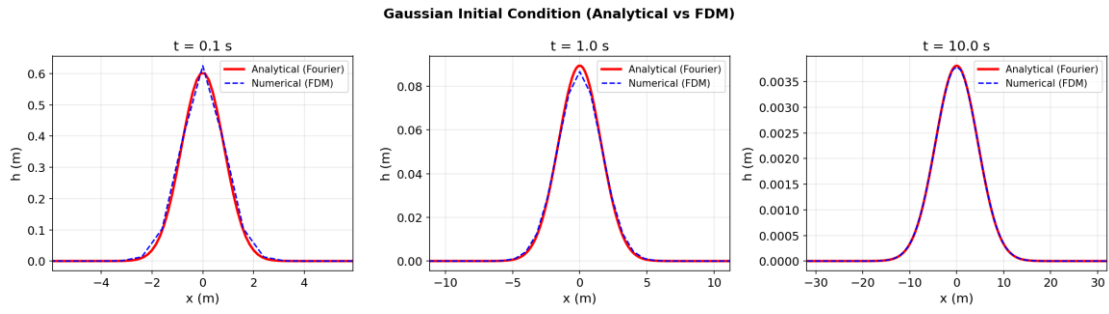


Figure 3: Comparison between the analytical and the numerical hydraulic head profiles for the Gaussian initial condition at $t = 0.1 \text{ s}$, 1.0 s , and 10.0 s

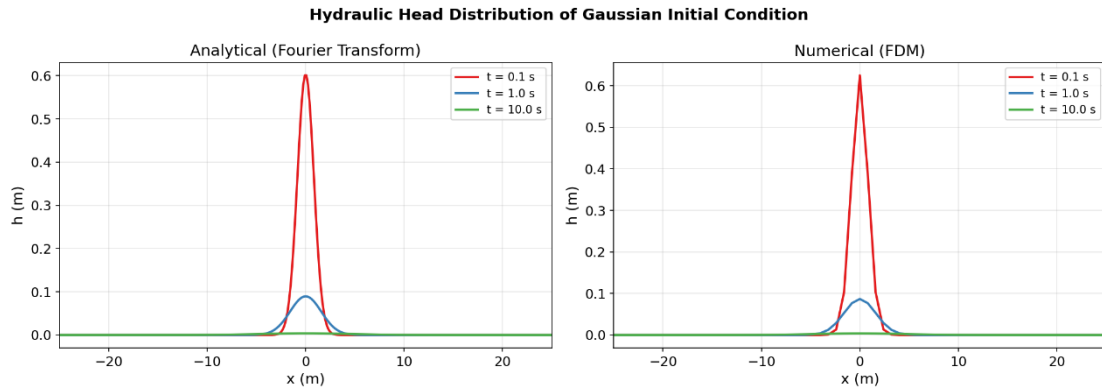


Figure 4: Diffusion of hydraulic head profiles comparing the analytical solution and numerical solution at $t = 0.1 \text{ s}$, 1.0 s , and 10.0 s

5.3. Grid Convergence Study

To verify the numerical implementation and to quantify the dependence of the L^2 error on grid resolution, a grid convergence study is carried out. The FDM is computed at ten grid resolutions, $N \in \{31, 51, 71, 91, 111, 131, 151, 171, 191, 211\}$, corresponding to grid

spacings ranging from $\Delta x = 2.6667$ m to $\Delta x = 0.3810$ m. The FTCS scheme has a theoretical truncation error of $O(\Delta t) + O(\Delta x^2)$ (Morton & Mayers 2005). Since the stability condition Eq. (47) constrains $\Delta t \propto \Delta x^2$, the overall error is expected to be $O(\Delta x^2)$ which yields a convergence rate of $p \approx 2$.

Figure 5 shows the L^2 error as a function of Δx on a log-log scale for the Gaussian initial condition at all three evaluation times. At $t = 10.0$ s, the curve closely follows the reference slope over finer grid resolutions, which confirms second-order spatial convergence. At earlier times, $t = 0.1$ s and $t = 1.0$ s, the errors are larger, but the error still decreases monotonically as the grid is refined.

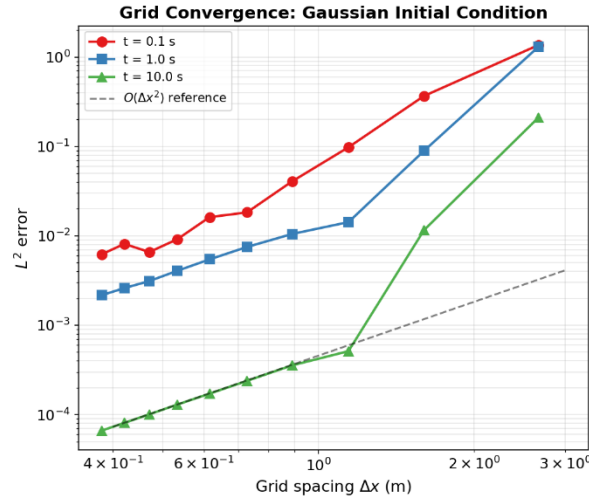


Figure 5: Grid convergence study for the 3D groundwater flow equation showing L^2 error versus grid spacing Δx for the Gaussian initial conditions

Table 2 presents the convergence rates estimated from consecutive grid pairs. The convergence rate p is estimated using the relation $p = \ln(L_1^2 / L_2^2) / \ln(\Delta x_1 / \Delta x_2)$. At coarser resolutions, $N = 31$ to $N = 71$, the estimated rates exceed the expected convergence rate $p \approx 2$. This means that the grid is too coarse to adequately resolve the spatial structure of the solution, and the error is dominated by resolution inadequacy. As the grid becomes finer, the convergence rate stabilizes into the theoretical $O(\Delta x^2)$ rate. From $N = 111$ onwards, the estimated convergence rate stabilises consistently near $p \approx 2$ with values ranging from 1.9576 to 2.0100. This confirms that the FDM implementation is correct and that the numerical scheme achieves its theoretical second-order convergence rate.

Table 2: Grid convergence results for the Gaussian initial condition at $t = 10.0$ s

N	Δx (m)	L^2 Error	Convergence Rate p
31	2.6667	2.1134×10^{-1}	—
51	1.6000	1.1570×10^{-2}	5.6869
71	1.1429	5.1233×10^{-4}	9.2645
91	0.8889	3.5529×10^{-4}	1.4565
111	0.7273	2.3839×10^{-4}	1.9885
131	0.6154	1.7147×10^{-4}	1.9725
151	0.5333	1.2957×10^{-4}	1.9576
171	0.4706	1.0075×10^{-4}	2.0100
191	0.4211	8.0767×10^{-5}	1.9878
211	0.3810	6.6289×10^{-5}	1.9739

6. Conclusion

This paper presents an analytical approach to solving three-dimensional transient groundwater flow equations using the Fourier transform method. This work extends the existing framework by deriving solutions for the Dirac Delta distribution and the Gaussian distribution, which provides a benchmark for analysing transient perturbations in aquifer systems.

The mathematical model is derived from Darcy's Law, and the continuity equations refer to the three-dimensional transient groundwater flow in a homogeneous, isotropic, and infinite domain. The Fourier transform simplifies the PDE by converting the spatial derivative into algebraic terms, transforming it into an ODE in the frequency domain. The inverse Fourier transform is then applied to convert the solution into the spatial domain. The analytical solution to the Gaussian initial condition was validated against a three-dimensional FDM using the FTCS scheme. Errors are quantified using the L^2 norm over the three-dimensional computational domain. Key findings include:

- (1) The analytical solution to the Dirac Delta initial condition corresponds to the fundamental solution to the diffusion equation. This solution describes the system's response to an instantaneous point perturbation at the origin, with the peak decaying as $(4\pi\alpha t)^{-3/2}$.
- (2) The Gaussian distribution solution maintains the Gaussian form while spreading over time, with the peak decaying as $(\sigma^2/\sigma^2 + 4\alpha t)^{3/2}$. The numerical validation shows good agreement between the analytical solution and FDM at all evaluation times. There is a slight overprediction at early times, which is consistent with the known behaviour of explicit finite difference schemes applied to diffusion problems.
- (3) The grid convergence study confirms that the FDM implementation exhibits the expected second-order spatial convergence rate. At $t = 10.0$ s, the estimated convergence rate stabilises to $p \approx 2$. This verifies both the implementation and the reliability of the analytical benchmark.
- (4) The analytical framework offers a substantial computational advantage over the numerical method. The closed-form solution can be evaluated directly at any point in space and time without iterative time-stepping. In contrast, the FDM requires thousands of explicit time steps constrained by the stability condition, which leads to a computational cost that depends on grid resolution and simulation duration.

The results derived in this work demonstrate that the analytical solutions provide exact benchmarks for three-dimensional transient groundwater flow under idealised conditions. These benchmarks serve two purposes. First, they provide a rigorous validation tool for numerical schemes, which enables the detection and quantification of discretisation errors. Second, and more broadly, they establish a stable analytical baseline by which additional physical complexities may be introduced. In particular, the deterministic solutions derived here form the foundation for stochastic extensions in which the hydraulic conductivity is treated as a spatially and temporally varying random field, driven by monsoon seasonality or other environmental forcing. By first establishing the exact behaviour of the deterministic system, any deviations observed in the stochastic model can be attributed to the environmental dynamics rather than to numerical instability.

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Appendix A. Evaluation of the Separable Ordinary Differential Equation

Equation (18) is a first-order linear ODE with constant coefficients,

$$\frac{\partial \hat{h}}{\partial t} = -\frac{k}{S_s}(a_x^2 + a_y^2 + a_z^2)\hat{h}(a_x, a_y, a_z, t). \quad (\text{A.1})$$

Let $\lambda = k/S_s(a_x^2 + a_y^2 + a_z^2)$. Then,

$$\frac{d\hat{h}}{\hat{h}} = -\lambda dt. \quad (\text{A.2})$$

Integrating both sides,

$$\ln|\hat{h}| = -\lambda t + C \quad (\text{A.3})$$

where C is a constant. Exponentiating Eq. (A.3) yields,

$$\hat{h}(a_x, a_y, a_z, t) = A \exp\left\{-\frac{k}{S_s}(a_x^2 + a_y^2 + a_z^2)t\right\}. \quad (\text{A.4})$$

Appendix B. Evaluation of the Gaussian Convolution Integral

The aim is to evaluate the integral,

$$I(x) = \int_{-\infty}^{\infty} \exp\left\{-\frac{x'^2}{\sigma^2} - \frac{(x-x')^2}{4\alpha t}\right\} dx', \quad (\text{B.1})$$

which comes from the convolution of two Gaussian functions in one dimension. To solve this, the exponents are combined into a single quadratic form,

$$-\frac{x'^2}{\sigma^2} - \frac{(x-x')^2}{4\alpha t} = -\left(\frac{1}{\sigma^2} + \frac{1}{4\alpha t}\right)x'^2 + \frac{2xx'}{4\alpha t} - \frac{x^2}{4\alpha t}. \quad (\text{B.2})$$

Let,

$$p = \frac{1}{\sigma^2} + \frac{1}{4\alpha t}, q = \frac{x}{2\alpha t}, r = \frac{x^2}{4\alpha t}. \quad (\text{B.3})$$

Rewrite the quadratic equations and perform completing the square,

$$-px'^2 + qx' - r = -p\left(x'^2 - \frac{q}{p}x'\right) - r = -p\left(x' - \frac{q}{2p}\right)^2 + \frac{q^2}{4p} - r. \quad (\text{B.4})$$

The integral takes the form,

$$I(x) = \exp\left\{\frac{q^2}{4p} - r\right\} \int_{-\infty}^{\infty} \exp\left\{-p\left(x' - \frac{q}{2p}\right)^2\right\} dx'. \quad (\text{B.5})$$

which is similar to the standard Gaussian integral,

$$\int_{-\infty}^{\infty} \exp\{-pu^2\} du = \sqrt{\frac{\pi}{p}}, \quad (\text{B.6})$$

where $u = x' - q/2p$. Thus,

$$I(x) = \exp\left\{\frac{q^2}{4p} - r\right\} \sqrt{\frac{\pi}{p}}. \quad (\text{B.7})$$

Compute,

$$\frac{q^2}{4p} - r = \frac{\left(\frac{x}{2\alpha t}\right)}{4\left(\frac{1}{\sigma^2} + \frac{1}{4\alpha t}\right)} - \frac{x^2}{4\alpha t} = \frac{x^2}{16\alpha^2 t^2 p} - \frac{x^2}{4\alpha t} = \frac{x^2 \sigma^2}{4\alpha t(4\alpha t + \sigma^2)} - \frac{x^2}{4\alpha t} = -\frac{x^2}{4\alpha t + \sigma^2}, \quad (\text{B.8})$$

and then,

$$\sqrt{\frac{\pi}{p}} = \sqrt{\frac{\pi}{\frac{1}{\sigma^2} + \frac{1}{4\alpha t}}} = \sqrt{\pi \frac{4\alpha t \sigma^2}{4\alpha t + \sigma^2}}. \quad (\text{B.9})$$

The final one-dimensional solution as given in Eq. (44),

$$I(x) = \exp\left\{-\frac{x^2}{4\alpha t + \sigma^2}\right\} \sqrt{\pi \frac{4\alpha t \sigma^2}{4\alpha t + \sigma^2}}. \quad (\text{B.10})$$

Extending to y- and z- directions, we have,

$$I(y) = \exp\left\{-\frac{y^2}{4\alpha t + \sigma^2}\right\} \sqrt{\pi \frac{4\alpha t \sigma^2}{4\alpha t + \sigma^2}} \text{ and } I(z) = \exp\left\{-\frac{z^2}{4\alpha t + \sigma^2}\right\} \sqrt{\pi \frac{4\alpha t \sigma^2}{4\alpha t + \sigma^2}}. \quad (\text{B.11})$$

Since the integral separates in three dimensions, the full solution is,

$$h(x, y, z, t) = \frac{A}{(4\pi\alpha t)^{3/2}} [I(x) \cdot I(y) \cdot I(z)]. \quad (\text{B.12})$$

Substituting Eqs. (B.10) and (B.11) gives,

$$h(x, y, z, t) = \frac{A}{(4\pi\alpha t)^{3/2}} \exp\left\{\frac{-x^2 - y^2 - z^2}{4\alpha t + \sigma^2}\right\} \left(\pi \frac{4\alpha t \sigma^2}{4\alpha t + \sigma^2}\right)^{\frac{3}{2}}. \quad (\text{B.13})$$

This is the same result as Eq. (43), which completes the derivation for the Gaussian initial condition.

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