

THE MOORE-PENROSE INVERSE MATRIX OF THE NON-NORMAL CYCLIC SUBGROUP GRAPH BY USING RANK FACTORIZATION

(Matriks Songsang Moore-Penrose bagi Graf Subkumpulan Kitaran tak Normal Menggunakan Pemfaktoran Pangkat)

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ABSTRACT

The inverse matrix can be determined for any non-singular or square matrix and it is particularly useful in solving linear systems. At the same time, the Moore-Penrose inverse matrix is an inverse matrix for any rectangular matrix constructed when the linear system has more unknowns than the number of values provided by the measurements. In previous studies, this inverse matrix has been applied to some graphs unrelated to finite groups. The main objective of this paper is to determine the Moore-Penrose inverse matrix in general form for the non-normal cyclic subgroup graph of two finite groups, namely the dihedral groups and the generalized quaternion groups. The digraph for these two finite groups represented by the incidence matrix is then applied to generate the Moore-Penrose inverse matrix using the rank factorization method. It is useful when the matrix is not of full rank. The Moore-Penrose inverse matrices are given in the form of matrix size, and the entries of the matrices are block matrices, as the digraph is disconnected and the incidence matrix is not unique.

Keywords: Moore-Penrose inverse matrix; subgroup graph; non-normal subgroup

ABSTRAK

Matriks songsang boleh ditentukan bagi sebarang matriks bukan singular atau matriks segi empat sama dan ia amat berguna dalam penyelesaian sistem linear. Pada masa yang sama, matriks songsang Moore-Penrose ialah matriks songsang bagi sebarang matriks segi empat tepat yang terhasil apabila sistem linear mempunyai pemboleh ubah melebihi bilangan nilai yang diperolehi daripada ukuran. Dalam kajian terdahulu, matriks songsang ini telah digunakan pada beberapa graf yang tidak berkaitan dengan kumpulan terhingga. Objektif utama kertas ini adalah untuk menentukan bentuk umum matriks songsang Moore-Penrose bagi graf subkumpulan kitaran tak normal bagi dua kumpulan terhingga, iaitu kumpulan dwihedron dan kumpulan kuaternion teritlak. Graf terarah bagi dua kumpulan terhingga ini yang diwakili oleh matriks insidens kemudiannya digunakan untuk menghasilkan matriks songsang Moore-Penrose menggunakan kaedah pemfaktoran pangkat penuh. Kaedah ini berguna apabila matriks tersebut tidak berpangkat penuh. Matriks songsang Moore-Penrose diberikan dalam bentuk saiz matriks dan entri-entri dalam matriks tersebut adalah matriks blok memandangkan graf terarah tersebut terpisah dan matriks insidens yang digunakan tidak bersifat unik.

Kata kunci: matriks songsang Moore-Penrose; graf subkumpulan; subkumpulan tidak normal

1. Introduction

Inverse matrices play a crucial role in solving linear systems and have numerous applications in various mathematical, physical and engineering fields. For non-singular square matrices, the inverse is well-defined and provides a direct solution to linear equations. However, the concept of an inverse must be generalized for rectangular or singular matrices. The Moore-Penrose

inverse, well known as the pseudoinverse, extends the idea of an inverse to such matrices, offering a powerful tool for solving underdetermined or overdetermined systems, particularly in cases where the linear system has more unknowns than the number of values provided by the measurements. The matrix is referred to as the Moore–Penrose inverse, as it was originally introduced by Moore (1920) and subsequently generalized by Penrose (1955). Miljković *et al.* (2012) solved a matrix equation using the Moore–Penrose inverse matrix, where a mathematical model of a uniform linear motion blur is represented by this equation. The application of this matrix in linear systems of algebraic equations has been discussed, and a numerical example is given by Kanan *et al.* (2019). In this paper, the Moore–Penrose inverse is later written as the *MP* inverse.

In algebraic graph theory, the study of graphs associated with finite groups has garnered significant attention due to their rich structural properties and wide-ranging applications in group theory, combinatorics, and related fields. Among these, the subgroup graph for non-normal subgroups, as defined by Rahin *et al.* (2020), provides a visual and combinatorial representation of the subgroup structure of a finite group. These graphs, when represented as incidence matrices, offer a bridge between group theory and linear algebra, enabling the application of matrix-theoretic techniques to solve problems in group theory. The issues in any field that has a similar model to the graph in this paper can be solved by using the *MP* inverse. Referring to Bhadra *et al.* (2021), the *MP* inverse is utilized to address hallucinations in Tomographic Image Reconstruction.

The incidence matrix, which represents a semigraph uniquely, was defined by Deshpande *et al.* (2015). This matrix is defined as an $n \times m$ matrix where the entries are 0, 1 or -1 . Another matrix that is used to represent a graph is the detour matrix, which is applied to find the energy of a graph. Abdussakir *et al.* (2018) applied the detour matrix to determine the spectrum and the energy of the conjugate graph complement for the dihedral group. The *MP* inverse matrix was discovered using the incidence matrix of a graph in 1997, where Bapat (1997) studied a theoretical description of the graph of the *MP* inverse of a tree. Azimi and Bapat (2018) computed the inverse matrix for a distance-regular graph. The latest research on the *MP* inverse matrix associated with the graph is given by Alazemi *et al.* (2023) who provide a relation between the *MP* inverse of the Laplacian and signless Laplacian matrices of a bipartite graph. This study focuses on two important classes of finite groups: dihedral groups and generalized quaternion groups. These groups are widely studied in group theory due to their unique properties and can be expressed using the following group representation:

$$\begin{aligned} D_{2n} &\cong \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle, n \geq 3, n \in \mathbb{N}, \\ Q_{4n} &\cong \langle a, b | a^{2n} = b^4 = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle, n \geq 2, n \in \mathbb{N}. \end{aligned}$$

The digraphs in this paper of these groups are particularly interesting because they often result in disconnected graphs, leading to non-unique incidence matrices. The following are the definitions and theorems that are related to determining the *MP* inverse matrix.

Theorem 1.1. (Rahin *et al.* 2025) *Let D_{2n} be the dihedral group of order $2n$, and H be a cyclic and non-normal subgroup of D_{2n} . Then, the non-normal cyclic subgroup graph for H is*

$$\Gamma_H^{NN}(D_{2n}) = \begin{cases} \left\{ \bigcup_{i=1}^k \Gamma_H(D_{2n}) \right\} \cup K_2, \text{ a digraph with } 3n - 1 \text{ edges,} & n = 2k + 1, k \in \mathbb{Z}^+, \\ \left\{ \bigcup_{i=1}^k \Gamma_H(D_{2n}) \right\} \cup K_2 \cup K_2, \text{ a digraph with } 3n - 2 \text{ edges,} & n = 2k + 2, k \in \mathbb{Z}^+, \end{cases}$$

where $\Gamma_H(D_{2n})$ is given as follows:

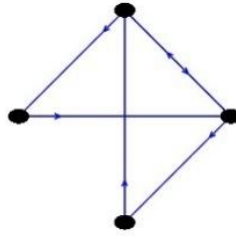


Figure 1: $\Gamma_H(D_{2n})$

Theorem 1.2. (Rahin et al. 2025) The incidence matrix of the non-normal cyclic subgroup graph for D_{2n} , $Q(\Gamma_H^{NN}(D_{2n}))$, where n is an odd integer, $n = 2k + 1, k \in \mathbb{Z}^+$, is a $2n \times (3n - 1)$ matrix as follows:

$$\begin{bmatrix} Q(\Gamma_H(D_{2n})) & 0 & \dots & \dots & 0 \\ 0 & Q(\Gamma_H(D_{2n})) & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & Q(\Gamma_H(D_{2n})) & \vdots \\ 0 & \dots & \dots & 0 & Q(K_2) \end{bmatrix},$$

where there are k copies of $Q(\Gamma_H(D_{2n}))$ in which

$$Q(\Gamma_H(D_{2n})) = \begin{bmatrix} -1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \text{ and } Q(K_2) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Theorem 1.3. (Rahin et al. 2025) The incidence matrix of the non-normal cyclic subgroup graph for D_{2n} , $Q(\Gamma_H^{NN}(D_{2n}))$ where n is an even integer, $n = 2k + 2, k \in \mathbb{Z}^+$, is a $2n \times 3n - 2$ matrix as follows:

$$\begin{bmatrix} Q(\Gamma_H(D_{2n})) & 0 & \dots & \dots & 0 \\ 0 & \ddots & 0 & \dots & \vdots \\ \vdots & 0 & Q(\Gamma_H(D_{2n})) & 0 & \vdots \\ \vdots & \vdots & 0 & Q(K_2) & \vdots \\ 0 & \dots & \dots & 0 & Q(K_2) \end{bmatrix},$$

where there are k copies of $Q(\Gamma_H(D_{2n}))$ in which

$$Q(\Gamma_H(D_{2n})) = \begin{bmatrix} -1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \text{ and } Q(K_2) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

From this theorem, the component of the disconnected graph for $\Gamma_H^{NN}(D_{2n})$ when n is even and odd is identified in the following corollary.

Corollary 1.1. (Rahin et al. 2025) Let D_{2n} be the dihedral group of order $2n$ where n is an odd integer, $n = 2k + 1, k \in \mathbb{Z}^+$ and H be a non-normal cyclic subgroup of D_{2n} . Then, the non-normal cyclic subgroup graph of D_{2n} has $k + 1$ components.

Corollary 1.2. (Rahin et al. 2025) Let D_{2n} be the dihedral group of order $2n$ where n is an even integer, $n = 2k + 2, k \in \mathbb{Z}^+$ and H be a non-normal cyclic subgroup of D_{2n} . Then, the non-normal cyclic subgroup graph of D_{2n} has $k + 2$ components.

Next, theorems are provided for the generalized quaternion groups.

Theorem 1.4. (Rahin et al. 2024) Let Q_{4n} be the generalized quaternion group,

$$Q_{4n} = \{1, a, a^2, \dots, a^{2n-1}, b, ab, a^2b, \dots, a^{2n-1}b\},$$

where n is an odd integer, $n = 2k + 1, k \in \mathbb{Z}^+$ and H_i is a non-normal cyclic subgroup of Q_{4n} where $H_i = \langle a^i b \rangle = \{1, a^i b, a^n, a^{n+i}b\}, 0 \leq i \leq 2n - 1$. Then, the non-normal cyclic subgroup graph for H_i is a digraph, $\Gamma_{H_i}^{NN}(Q_{4n}) = \{\cup_{i=1}^k \Gamma_H(Q_{4n})\} \cup K_4$ with $14n - 2$ edges where $\Gamma_H(Q_{4n})$ is as follows:

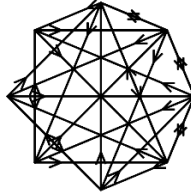


Figure 2: $\Gamma_H(Q_{4n})$ where n is odd

Theorem 1.5. (Rahin et al. 2024) Let Q_{4n} be the generalized quaternion group,

$$Q_{4n} = \{1, a, a^2, \dots, a^{2n-1}, b, ab, a^2b, \dots, a^{2n-1}b\},$$

where n is an even integer, $n = 2k + 2, k \in \mathbb{Z}^+$ and H is a non-normal subgroup cyclic of G with $H_i = \langle a^i b \rangle = \{1, a^i b, a^n, a^{n+i}b\}, 0 \leq i \leq 2n - 1$. Then, the non-normal cyclic subgroup graph for H is a digraph, $\Gamma_H^{NN}(Q_{4n}) = \{\cup_{i=1}^k \Gamma_H(Q_{4n})\} \cup K_4 \cup K_4$ with $14n - 4$ edges, where the pattern of $\Gamma_H(Q_{4n})$ is as in Figure 2.

The following theorems are provided for the incidence matrix of the graph discussed.

Theorem 1.6. (Rahin et al. 2024) The incidence matrix of the non-normal cyclic subgroup graph, $Q(\Gamma_{H_i}^{NN}(Q_{4n}))$ where n is an odd integer, $n = 2k + 1, k \in \mathbb{Z}^+$, is a $4n \times (14n - 2)$ matrix as follows:

$$Q(\Gamma_{H_i}^{NN}(Q_{4n})) = \begin{bmatrix} Q(\Gamma_H(Q_{4n})) & 0 & \dots & \dots & 0 \\ 0 & Q(\Gamma_H(Q_{4n})) & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & Q(\Gamma_H(Q_{4n})) & \vdots \\ 0 & \dots & \dots & 0 & Q(K_4) \end{bmatrix},$$

where there are k copies of $Q(\Gamma_H(Q_{4n}))$, the incidence matrix of $\Gamma_H(Q_{4n})$, and $Q(K_4)$ is the incidence matrix of a complete digraph with four vertices.

Theorem 1.7. (Rahin et al. 2024) The incidence matrix of the non-normal cyclic subgroup graph, where n is an even integer, $n = 2k + 2, k \in \mathbb{Z}^+$, is a $4n \times (14n - 4)$ matrix as follows:

$$Q(\Gamma_{H_i}^{NN}(Q_{4n})) = \begin{bmatrix} Q(\Gamma_H(Q_{4n})) & 0 & \dots & \dots & 0 \\ 0 & \ddots & 0 & \dots & \vdots \\ \vdots & 0 & Q(\Gamma_H(Q_{4n})) & 0 & \vdots \\ \vdots & \vdots & 0 & Q(K_4) & \vdots \\ 0 & \dots & \dots & 0 & Q(K_4) \end{bmatrix},$$

where there are k copies of $Q(\Gamma_H(Q_{4n}))$, the incidence matrix of $\Gamma_H(Q_{4n})$, and $Q(K_4)$ is the incidence matrix of a complete digraph with four vertices.

The matrix labelled with 0 in the theorems before is the zero matrix. An example of the incidence matrix is visualized as follows.

The non-normal cyclic subgroup graph of D_{10} when $H_1 = \langle b \rangle = \{1, b\}$ is a directed graph with 14 directed edges, $\Gamma_{H_1}^{NN}(D_{10}) = \Gamma_H(D_{10}) \cup \Gamma_H(D_{10}) \cup K_2$. Since the graph is a disconnected graph, then

$$Q(\Gamma_H(D_{10})) = \begin{bmatrix} -1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

is the incidence matrix of $\Gamma_H(D_{10})$ and

$$Q(K_2) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

is for the complete digraph with two vertices. Then, the incidence matrix of D_{10} , $Q(\Gamma_H^{NN}(D_{10}))$ is a 10×14 matrix as follows:

$$\begin{bmatrix} Q(\Gamma_H(D_{10})) & 0 & 0 \\ 0 & Q(\Gamma_H(D_{10})) & 0 \\ 0 & 0 & Q(K_2) \end{bmatrix}.$$

Rank factorization is a robust technique for computing the *MP* inverse, especially for matrices that are not of full rank. The following theorem and definitions are used in computing the *MP* inverse.

Theorem 1.8. (Bapat 2010) *If $\Gamma(G)$ is a graph on m vertices with p connected components, then the rank of the incidence matrix, $r = m - p$.*

Definition 1.1. (Piziak and Odell 1999) Let A be an $m \times n$ matrix of rank $r > 0$. A full rank factorization of A exists if there are $m \times r$ matrix F and an $r \times n$ matrix G such that $A = FG$.

Definition 1.2. (Penrose 1955) Let A be an $m \times n$ matrix, then a unique *MP* inverse $n \times m$ matrix A^+ exists such that it satisfies the following equations:

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad (AA^+)^T = AA^+, \quad (A^+A)^T = A^+A$$

where A^T denotes the transpose of the matrix A .

The incidence matrix of a graph may be represented through a full-rank factorization upon transformation into reduced row echelon form, as defined in Definition 1.1. Consequently, the following proposition is established to derive the Moore–Penrose inverse of the incidence matrix.

Proposition 1.1. (Bapat 2010) *Let $Q = BC$ be a rank factorization where $B^+ = (B^TB)^{-1}B^T$ and $C^+ = C^T(CC^T)^{-1}$. Then, the *MP* inverse matrix of the incidence matrix Q is $Q^+ = C^+B^+$.*

This work aims to derive the *MP* inverse matrix for these graphs using the rank factorization method. This work opens new avenues for exploring the interplay among group theory, graph theory and linear algebra, with potential applications in coding theory, cryptography and network analysis.

2. Main Results

This section presents the key findings of the graphs addressed in this study for the two finite groups mentioned before, for the cases of n being an odd and even positive integer, respectively.

2.1. The *MP* Inverse of Dihedral Groups

The following lemmas are given for the incidence matrix of the graph to be written in a full rank factorization after transforming the incidence matrix to a reduced row echelon form. The full rank factorization of the incidence matrix is given in terms of matrix size before constructing the *MP* inverse matrix in this research.

Lemma 2.1.1. Let $Q(\Gamma_H^{NN}(D_{2n}))$ be the incidence matrix of the non-normal cyclic subgroup graph for D_{2n} with matrix size $2n \times (3n - 1)$ where n is an odd integer, $n = 2k + 1, k \in \mathbb{Z}^+$, then there are $2n \times r$ matrix B and $r \times 3n - 1$ matrix C such that

$$Q(\Gamma_H^{NN}(D_{2n})) = B_{2n \times r} C_{r \times (3n-1)}$$

is a full rank factorization of $Q(\Gamma_H^{NN}(D_{2n}))$, where $r = 2n - k - 1$.

Proof. Suppose D_{2n} is the dihedral group where n is an odd integer and $\Gamma_H^{NN}(D_{2n})$ is its non-normal cyclic subgroup graph. From Theorem 1.2, the incidence matrix, $Q(\Gamma_H^{NN}(D_{2n}))$ is a $2n \times (3n - 1)$ and from Theorem 1.1 and Corollary 1.1, the $\Gamma_H^{NN}(D_{2n})$ graph has $k + 1$ components. Since the number of vertices of the graph is $2n$, then the rank of the incidence matrix is $r = m - p = 2n - (k + 1) = 2n - k - 1$ based on Theorem 1.8. After transforming the incidence matrix into its reduced row echelon form, a $2n \times r$ matrix B and an $r \times 3n - 1$ matrix C are obtained. The matrix B is obtained by removing the column of the incidence matrix, which depends on the non-pivot column of the incidence matrix, in a reduced row echelon form. Meanwhile, matrix C is the matrix obtained after transforming the incidence matrix to a reduced row echelon form, deleting the zero rows. By Definition 1.1, the full rank factorization of $Q(\Gamma_H^{NN}(D_{2n}))$ exists and can be expressed as

$$Q(\Gamma_H^{NN}(D_{2n})) = B_{2n \times r} C_{r \times (3n-1)}, \text{ where } r = 2n - k - 1. \quad \square$$

This lemma is then applied to obtain the MP inverse matrix of the graph, which is denoted as Q^+ . From Proposition 1.1, the MP inverse matrix of the incidence matrix, Q is $Q^+ = C^+ B^+$. Thus, an inverse and a transpose matrix of matrix B and matrix C are needed to get B^+ and C^+ . The following lemma is given for getting the matrix size for B^+ and C^+ .

Lemma 2.1.2. Let $Q(\Gamma_H^{NN}(D_{2n})) = B_{2n \times r} C_{r \times (3n-1)}$ where n is an odd integer, $n = 2k + 1, k \in \mathbb{Z}^+$ and $r = 2n - k - 1$. Then, the size of the MP inverse of B and C , B^+ and C^+ are $r \times 2n$ and $(3n - 1) \times r$, respectively.

Proof. Suppose D_{2n} is the dihedral group where n is an odd integer and $\Gamma_H^{NN}(D_{2n})$ is its non-normal cyclic subgroup graph. Let $Q(\Gamma_H^{NN}(D_{2n})) = B_{2n \times r} C_{r \times (3n-1)}$ where $r = 2n - k - 1$ be the full rank factorization of the incidence matrix of the non-normal cyclic subgroup graph for D_{2n} . Thus, the size of the matrix B^+ and matrix C^+ are $r \times 2n$ and $(3n - 1) \times r$ since $B^+ = (B^T B)^{-1} B^T$ and $C^+ = C^T (C C^T)^{-1}$ are the MP inverse matrix of matrix B and matrix C , respectively. $\square$

The next lemmas are given for even cases.

Lemma 2.1.3. Let $Q(\Gamma_H^{NN}(D_{2n}))$ be the incidence matrix of the non-normal cyclic subgroup graph for D_{2n} with matrix size $2n \times (3n - 2)$ where n is an even integer, $n = 2k + 2, k \in \mathbb{Z}^+$. Then there are $2n \times r$ matrix B and an $r \times 3n - 2$ matrix C such that

$$Q\left(\Gamma_H^{NN}(D_{2n})\right) = B_{2n \times r} C_{r \times (3n-2)}$$

is a full rank factorization of $Q\left(\Gamma_H^{NN}(D_{2n})\right)$, where $r = 2n - k - 2$.

Proof. Suppose D_{2n} is the dihedral group where n is an even integer and $\Gamma_H^{NN}(D_{2n})$ is its non-normal subgroup graph. From Theorem 1.3, the incidence matrix, $Q\left(\Gamma_H^{NN}(D_{2n})\right)$ is a $2n \times (3n - 2)$ and from Theorem 1.1 and Corollary 1.2, the $\Gamma_H^{NN}(D_{2n})$ graph $\Gamma_H^{NN}(D_{2n})$ has $k + 2$ components. Thus, the rank of the incidence matrix is $r = 2n - k - 2$ based on Theorem 1.8. After transforming the incidence matrix into its reduced row echelon form, a $2n \times r$ matrix B and an $r \times 3n - 2$ matrix C are obtained. The matrix B is obtained by removing the column of the incidence matrix, which depends on the non-pivot column of the incidence matrix, in a reduced row echelon form. Meanwhile, matrix C is the matrix obtained after transforming the incidence matrix to a reduced row echelon form, deleting the zero rows. By Definition 1.1, the full rank factorization of $Q\left(\Gamma_H^{NN}(D_{2n})\right)$ exists and can be expressed as

$$Q\left(\Gamma_H^{NN}(D_{2n})\right) = B_{2n \times r} C_{r \times (3n-2)}, \text{ where } r = 2n - k - 2. \quad \square$$

Lemma 2.1.4. Let $Q\left(\Gamma_H^{NN}(D_{2n})\right) = B_{2n \times r} C_{r \times (3n-2)}$ be the incidence matrix of the non-normal cyclic subgroup graph for D_{2n} where n is an even integer, $n = 2k + 2, k \in \mathbb{Z}^+$ and $r = 2n - k - 2$. Then, the sizes of the MP inverse of B and C , B^+ and C^+ are $r \times 2n$ and $(3n - 2) \times r$, respectively.

Proof. Suppose D_{2n} is the dihedral group where n is an even integer and $\Gamma_H^{NN}(D_{2n})$ is its non-normal cyclic subgroup graph. Let $Q\left(\Gamma_H^{NN}(D_{2n})\right) = B_{2n \times r} C_{r \times (3n-2)}$ where $r = 2n - k - 2$ be the full rank factorization of the incidence matrix. Thus, the size of the matrices B^+ and C^+ are $r \times 2n$ and $(3n - 2) \times r$ since $B^+ = (B^T B)^{-1} B^T$ and $C^+ = C^T (C C^T)^{-1}$ are the MP inverse matrix of matrix B and matrix C , respectively. $\square$

Next is the theorem for constructing the general form of the MP inverse matrix of the dihedral groups.

Theorem 2.1.1. Let D_{2n} be the dihedral group of order $2n$ where n is an odd integer, $n = 2k + 1, k \in \mathbb{Z}^+$. Then, the MP inverse matrix of the incidence matrix of the non-normal cyclic subgroup graph for D_{2n} is a $(3n - 1) \times 2n$ matrix as follows:

$$Q^+\left(\Gamma_H^{NN}(D_{2n})\right) = \begin{bmatrix} Q^+(\Gamma_H(D_{2n})) & 0 & \dots & \dots & 0 \\ 0 & Q^+(\Gamma_H(D_{2n})) & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & Q^+(\Gamma_H(D_{2n})) & \vdots \\ 0 & \dots & \dots & 0 & Q^+(K_2) \end{bmatrix},$$

where there are k copies of $Q^+(\Gamma_H(D_{2n}))$.

Proof. From Theorem 1.2, $Q(\Gamma_H^{NN}(D_{2n}))$ is a $2n \times (3n - 1)$ matrix as follows:

$$\begin{bmatrix} Q(\Gamma_H(D_{2n})) & 0 & \cdots & \cdots & 0 \\ 0 & Q(\Gamma_H(D_{2n})) & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & Q(\Gamma_H(D_{2n})) & \vdots \\ 0 & \cdots & \cdots & 0 & Q(K_2) \end{bmatrix},$$

where there are k copies of $Q(\Gamma_H(D_{2n}))$. The full rank factorization is from Lemma 2.1.1 as follows:

$$Q(\Gamma_H^{NN}(D_{2n})) = B_{2n \times r} C_{r \times (3n-1)}, \quad r = 2n - k - 1$$

and from Proposition 1.1, the MP inverse matrix is obtained by rank factorization,

$$Q^+ = C^+ B^+.$$

By using Lemma 2.1.2, then

$$Q^+ = C^+_{(3n-1) \times r} B^+_{r \times 2n}, \quad r = 2n - k - 1.$$

Thus, the matrix size of the MP inverse matrix of $Q^+(\Gamma_H^{NN}(D_{2n}))$ is $(3n - 1) \times 2n$ and the MP inverse matrix can be written as

$$Q^+(\Gamma_H^{NN}(D_{2n})) = \begin{bmatrix} Q^+(\Gamma_H(D_{2n})) & 0 & \cdots & \cdots & 0 \\ 0 & Q^+(\Gamma_H(D_{2n})) & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & Q^+(\Gamma_H(D_{2n})) & \vdots \\ 0 & \cdots & \cdots & 0 & Q^+(K_2) \end{bmatrix},$$

where there are k copies of $Q^+(\Gamma_H(D_{2n}))$. $\square$

Theorem 2.1.2. Let D_{2n} be the dihedral group of order $2n$ where n is an even integer, $n = 2k + 2, k \in \mathbb{Z}^+$. Then, the MP inverse matrix of the incidence matrix of the non-normal cyclic subgroup graph for D_{2n} is $(3n - 2) \times 2n$ matrix as follows:

$$Q^+(\Gamma_H^{NN}(D_{2n})) = \begin{bmatrix} Q^+(\Gamma_H(D_{2n})) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & \vdots \\ \vdots & 0 & Q^+(\Gamma_H(D_{2n})) & 0 & \vdots \\ \vdots & \vdots & 0 & Q^+(K_2) & \vdots \\ 0 & \cdots & \cdots & 0 & Q^+(K_2) \end{bmatrix},$$

where there are k copies of $Q^+(\Gamma_H(D_{2n}))$.

Proof. From Theorem 1.3, $Q(\Gamma_H^{NN}(D_{2n}))$ is a $2n \times (3n - 2)$ matrix as follows:

$$\begin{bmatrix} Q(\Gamma_H(D_{2n})) & O & \dots & \dots & O \\ O & \ddots & O & \dots & \vdots \\ \vdots & O & Q(\Gamma_H(D_{2n})) & O & \vdots \\ \vdots & \vdots & O & Q(K_2) & \vdots \\ O & \dots & \dots & O & Q(K_2) \end{bmatrix},$$

where there are k copies of $Q(\Gamma_H(D_{2n}))$. The full rank factorization is from Lemma 2.1.3 as follows:

$$Q(\Gamma_H^{NN}(D_{2n})) = B_{2n \times r} C_{r \times (3n-2)}, r = 2n - k - 2$$

and from Proposition 1.1, the MP inverse matrix is obtained by rank factorization

$$Q^+ = C^+ B^+.$$

The matrix size of B^+ and C^+ is stated in Lemma 2.1.4. Thus,

$$Q^+ = C^+_{(3n-2) \times r} B^+_{r \times 2n}, \quad r = 2n - k - 2.$$

Therefore, the matrix size of $Q^+(\Gamma_H^{NN}(D_{2n}))$ is $(3n - 2) \times 2n$ and the MP inverse matrix can be written as

$$Q^+(\Gamma_H^{NN}(D_{2n})) = \begin{bmatrix} Q^+(\Gamma_H(D_{2n})) & O & \dots & \dots & O \\ O & \ddots & O & \dots & \vdots \\ \vdots & O & Q^+(\Gamma_H(D_{2n})) & O & \vdots \\ \vdots & \vdots & O & Q^+(K_2) & \vdots \\ O & \dots & \dots & O & Q^+(K_2) \end{bmatrix},$$

where there are k copies of $Q^+(\Gamma_H(D_{2n}))$. $\square$

An example is provided for the dihedral group of order six.

Example 2.1.1. Let D_6 be the dihedral group of order six and H be a non-normal cyclic subgroup of D_6 . The incidence matrix of the non-normal cyclic subgroup graph for the dihedral group of order six, D_6 is a 6×8 matrix as follows:

$$Q(\Gamma_H^{NN}(D_6)) = \begin{bmatrix} -1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

This incidence matrix is written in a reduced row echelon form before converting it into a full rank factorization as follows:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are two components of this graph. Thus, this incidence matrix can be written in a full rank factorization where $r = 2(3) - 2 = 4$. Thus, $Q\left(\Gamma_H^{NN}(D_{2n})\right) = B_{6 \times 4} C_{4 \times 8}$ and the matrix B and the matrix C are given as follows:

$$B = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

These two matrices, B and C are obtained from the incidence matrix, which is already in a reduced row echelon form. The matrix B is obtained by removing the column of the incidence matrix, $Q\left(\Gamma_H^{NN}(D_6)\right)$, which depends on the non-pivot column of the incidence matrix in a reduced row echelon form. Meanwhile, matrix C is the matrix obtained after transforming the incidence matrix to a reduced row echelon form, deleting the zero rows. An inverse and a transpose matrix of the matrix B and C are applied to get $B^+ = (B^T B)^{-1} B^T$ and $C^+ = C^T (C C^T)^{-1}$ which are given as follows:

$$B^+ = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

and

$$C^+ = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & 0 \\ \frac{1}{6} & \frac{7}{12} & -\frac{1}{12} & 0 \\ -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} & 0 \\ -\frac{1}{6} & -\frac{1}{12} & \frac{7}{12} & 0 \\ -\frac{1}{6} & -\frac{1}{12} & -\frac{5}{12} & 0 \\ \frac{1}{6} & -\frac{5}{12} & -\frac{1}{12} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

It can be seen that the matrix size of B^+ is 4×6 and the matrix size of C^+ is 8×4 , then the matrix size of Q^+ where

$$Q^+ = C^+_{8 \times 4} B^+_{4 \times 6}$$

is of size 8×6 as follows:

$$Q^+ \left(\Gamma_H^{NN}(D_6) \right) = \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\ -\frac{5}{24} & -\frac{1}{24} & -\frac{1}{8} & \frac{3}{8} & 0 & 0 \\ \frac{1}{6} & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ -\frac{1}{24} & -\frac{5}{24} & \frac{3}{8} & -\frac{1}{8} & 0 & 0 \\ \frac{5}{24} & \frac{1}{24} & -\frac{3}{8} & \frac{1}{8} & 0 & 0 \\ \frac{1}{24} & \frac{5}{24} & \frac{1}{8} & -\frac{3}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

2.2. The MP Inverse of Generalized Quaternion Groups

Similarly, in the previous subsection, the following lemmas are given before constructing the MP inverse matrix in this research.

Lemma 2.2.1. *Let $Q(\Gamma_{H_i}^{NN}(Q_{4n}))$ be the incidence matrix of the non-normal cyclic subgroup graph for Q_{4n} with matrix size $4n \times (14n - 2)$ where n is an odd integer, $n = 2k + 1$, $k \in \mathbb{Z}^+$. Then there are $4n \times r$ matrix B and $r \times 14n - 2$ matrix C such that*

$$Q(\Gamma_{H_i}^{NN}(Q_{4n})) = B_{4n \times r} C_{r \times (14n-2)}$$

is a full rank factorization of $Q(\Gamma_H^{NN}(D_{2n}))$, where $r = 4n - k - 1$.

Proof. Suppose Q_{4n} is the generalized quaternion group where n is an odd integer and $\Gamma_{H_i}^{NN}(Q_{4n})$ is its non-normal cyclic subgroup graph. From Theorem 1.6, the incidence matrix, $Q(\Gamma_{H_i}^{NN}(Q_{4n}))$ is a $4n \times (14n - 2)$ and from Theorem 1.4, $\Gamma_{H_i}^{NN}(Q_{4n})$ has $k + 1$ components since there are k components of $\Gamma_H(Q_{4n})$ and one complete digraph with four vertices. The number of vertices of the graph is $4n$, then the rank of the incidence matrix is $r = 4n - k - 1$ based on Theorem 1.8. After transforming the incidence matrix into its reduced row echelon form, a $4n \times r$ matrix B and an $r \times 14n - 2$ matrix C are obtained. The matrix B is obtained by removing the column of the incidence matrix, which depends on the non-pivot column of the incidence matrix, in a reduced row echelon form. Meanwhile, matrix C is the matrix obtained after transforming the incidence matrix to a reduced row echelon form, deleting the zero rows. By Definition 1.1, the full rank factorization of $Q(\Gamma_{H_i}^{NN}(Q_{4n}))$ exists and can be expressed as

$$Q(\Gamma_{H_i}^{NN}(Q_{4n})) = B_{4n \times r} C_{r \times (14n-2)}, \text{ where } r = 4n - k - 1. \quad \square$$

The incidence matrix is then applied to obtain the MP inverse matrix of the graph, which is denoted as Q^+ . Thus, the following lemma is given for getting the matrix size for B^+ and C^+ .

Lemma 2.2.2. *Let $Q(\Gamma_{H_i}^{NN}(Q_{4n})) = B_{4n \times r} C_{r \times (14n-2)}$ where n is an odd integer, $n = 2k + 1$, $k \in \mathbb{Z}^+$ and $r = 4n - k - 1$. Then, the sizes of the MP inverse of B and C , B^+ and C^+ are $r \times 4n$ and $(14n - 2) \times r$, respectively.*

Proof. Suppose Q_{4n} is the generalized quaternion group where n is an odd integer and $\Gamma_{H_i}^{NN}(Q_{4n})$ is its non-normal cyclic subgroup graph. Let $Q(\Gamma_{H_i}^{NN}(Q_{4n})) = B_{4n \times r} C_{r \times (14n-2)}$ where $r = 4n - k - 1$ be the full rank factorization of the incidence matrix of the non-normal cyclic subgroup graph for Q_{4n} . Thus, the size of the matrices of B^+ and C^+ are $r \times 4n$ and $(14n - 2) \times r$ since $B^+ = (B^T B)^{-1} B^T$ and $C^+ = C^T (C C^T)^{-1}$ are the MP inverse matrix of the matrices B and C , respectively. $\square$

The next lemmas are given when n is an even integer.

Lemma 2.2.3. Let $Q_{4n}Q(\Gamma_{H_i}^{NN}(Q_{4n}))$ be the incidence matrix of the non-normal cyclic subgroup graph for Q_{4n} with matrix size $4n \times (14n - 4)$ where n is an even integer, $n = 2k + 2, k \in \mathbb{Z}^+$. Then there are $4n \times r$ matrix B and $r \times 14n - 4$ matrix C such that

$$Q(\Gamma_{H_i}^{NN}(Q_{4n})) = B_{4n \times r} C_{r \times (14n-4)}$$

is a full rank factorization of $Q(\Gamma_H^{NN}(D_{2n}))$, where $r = 4n - k - 2$.

Proof. Suppose G is the generalized quaternion group, Q_{4n} where n is an even integer and $\Gamma_{H_i}^{NN}(Q_{4n})$ is its non-normal subgroup graph. From Theorem 1.7, the incidence matrix, $Q(\Gamma_{H_i}^{NN}(Q_{4n}))$ is a $4n \times (14n - 4)$ and from Theorem 1.6, $\Gamma_{H_i}^{NN}(Q_{4n})$ has $k + 2$ components since there are k components of $\Gamma_H(Q_{4n})$ and two complete digraphs with four vertices. The number of vertices of the graph is $4n$, then the rank of the incidence matrix is $r = 4n - k - 2$ based on Theorem 1.8. After transforming the incidence matrix into its reduced row echelon form, a $4n \times r$ matrix B and an $r \times 14n - 4$ matrix C are obtained. The matrix B is obtained by removing the column of the incidence matrix, which depends on the non-pivot column of the incidence matrix, in a reduced row echelon form. Meanwhile, matrix C is the matrix obtained after transforming the incidence matrix to a reduced row echelon form, deleting the zero rows. By Definition 1.1, the full rank factorization of $Q(\Gamma_{H_i}^{NN}(Q_{4n}))$ exists and can be expressed as

$$Q(\Gamma_{H_i}^{NN}(Q_{4n})) = B_{4n \times r} C_{r \times (14n-4)}, \text{ where } r = 4n - k - 2. \quad \square$$

Lemma 2.2.4. Let $Q(\Gamma_{H_i}^{NN}(Q_{4n})) = B_{4n \times r} C_{r \times (14n-4)}$ be the incidence matrix of the non-normal cyclic subgroup graph for Q_{4n} where n is an even integer, $n = 2k + 2, k \in \mathbb{Z}^+$ and $r = 4n - k - 2$. Then, the sizes of the MP inverse of B and C , B^+ and C^+ are $r \times 4n$ and $(14n - 4) \times r$, respectively.

Proof. Suppose G is the dihedral group, Q_{4n} where n is an even integer and $\Gamma_{H_i}^{NN}(Q_{4n})$ is its non-normal cyclic subgroup graph. Let $Q(\Gamma_{H_i}^{NN}(Q_{4n})) = B_{4n \times r} C_{r \times (14n-4)}$ where $r = 4n - 3$ be the full rank factorization of the matrix. Thus, the size of the matrices B^+ and C^+ are $r \times 4n$ and $(14n - 4) \times r$ respectively since $B^+ = (B^T B)^{-1} B^T$ and $C^+ = C^T (C C^T)^{-1}$ are the MP inverse matrix of the matrix B and matrix C , respectively. $\square$

These lemmas are applied in the following theorem to construct the general form of the MP inverse matrix for the generalized quaternion groups. The incidence matrix is then applied to obtain the MP inverse matrix of the graph, which is denoted as Q^+ .

Theorem 2.2.1. Let Q_{4n} be the generalized quaternion group of order $4n$ where n is an odd integer, $n = 2k + 1, k \in \mathbb{Z}^+$. Then, the MP inverse matrix of the incidence matrix of the non-normal cyclic subgroup graph for Q_{4n} , $Q^+(\Gamma_H^{NN}(Q_{4n}))$ is a $(14n - 2) \times 4n$ matrix as follows:

$$\begin{bmatrix} Q^+(\Gamma_H(Q_{4n})) & 0 & \cdots & \cdots & 0 \\ 0 & Q^+(\Gamma_H(Q_{4n})) & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & Q^+(\Gamma_H(Q_{4n})) & \vdots \\ 0 & \cdots & \cdots & 0 & Q^+(K_4) \end{bmatrix},$$

where there are k copies of $Q^+(\Gamma_H(Q_{4n}))$.

Proof. From Theorem 1.6, $Q^+(\Gamma_H^{NN}(Q_{4n}))$ is a $4n \times (14n - 2)$ matrix as follows:

$$\begin{bmatrix} Q(\Gamma_H(Q_{4n})) & 0 & \cdots & \cdots & 0 \\ 0 & Q(\Gamma_H(Q_{4n})) & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & Q(\Gamma_H(Q_{4n})) & \vdots \\ 0 & \cdots & \cdots & 0 & Q(K_4) \end{bmatrix},$$

where there are k copies of $Q(\Gamma_H(Q_{4n}))$. The full rank factorization is from Lemma 2.2.1 as follows:

$$Q(\Gamma_H^{NN}(Q_{4n})) = B_{4n \times r} C_{r \times (14n-2)}, \quad r = 4n - k - 1,$$

and from Proposition 1.1, the MP inverse matrix is obtained by rank factorization,

$$Q^+ = C^+ B^+.$$

By using Lemma 2.2.2, then

$$Q^+ = C^+_{(14n-2) \times r} B^+_{r \times 4n}, \quad r = 4n - k - 1.$$

Thus, the matrix size of the MP inverse matrix of $Q^+(\Gamma_H^{NN}(Q_{4n}))$ is $(14n - 2) \times 4n$ and the MP inverse matrix can be written as

$$\begin{bmatrix} Q^+(\Gamma_H(Q_{4n})) & 0 & \cdots & \cdots & 0 \\ 0 & Q^+(\Gamma_H(Q_{4n})) & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & Q^+(\Gamma_H(Q_{4n})) & \vdots \\ 0 & \cdots & \cdots & 0 & Q^+(K_4) \end{bmatrix},$$

where there are k copies of $Q^+(\Gamma_H(Q_{4n}))$. $\square$

Theorem 2.2.2. Let Q_{4n} be the generalized quaternion group of order $4n$ where n is an even integer, $n = 2k + 2, k \in \mathbb{Z}^+$. Then, the MP inverse matrix of the incidence matrix of the non-normal cyclic subgroup graph for Q_{4n} is a $(14n - 4) \times 4n$ matrix as follows:

$$Q^+ \left(\Gamma_{H_i}^{NN}(Q_{4n}) \right) = \begin{bmatrix} Q^+(\Gamma_H(Q_{4n})) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & \vdots \\ \vdots & 0 & Q^+(\Gamma_H(Q_{4n})) & 0 & \vdots \\ \vdots & \vdots & 0 & Q^+(K_4) & \vdots \\ 0 & \cdots & \cdots & 0 & Q^+(K_4) \end{bmatrix}.$$

Proof. From Theorem 1.7, $Q \left(\Gamma_{H_i}^{NN}(Q_{4n}) \right)$ is a $4n \times (14n - 4)$ matrix as follows:

$$\begin{bmatrix} Q(\Gamma_H(Q_{4n})) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & \vdots \\ \vdots & 0 & Q(\Gamma_H(Q_{4n})) & 0 & \vdots \\ \vdots & \vdots & 0 & Q(K_4) & \vdots \\ 0 & \cdots & \cdots & 0 & Q(K_4) \end{bmatrix}.$$

The full rank factorization is from Lemma 2.2.3 as follows:

$$Q \left(\Gamma_{H_i}^{NN}(Q_{4n}) \right) = B_{4n \times r} C_{r \times (14n-4)}, \quad r = 4n - k - 2,$$

and from Proposition 1.1, the MP inverse matrix is obtained by rank factorization,

$$Q^+ = C^+ B^+.$$

By using Lemma 2.2.4, then

$$Q^+ = C^+_{(14n-4) \times r} B^+_{r \times 4n}, \quad r = 4n - k - 2.$$

Therefore, the matrix size of Q^+ is $(14n - 4) \times 4n$ and the MP inverse matrix can be written as

$$Q^+ \left(\Gamma_{H_i}^{NN}(Q_{4n}) \right) = \begin{bmatrix} Q^+(\Gamma_H(Q_{4n})) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & \vdots \\ \vdots & 0 & Q^+(\Gamma_H(Q_{4n})) & 0 & \vdots \\ \vdots & \vdots & 0 & Q^+(K_4) & \vdots \\ 0 & \cdots & \cdots & 0 & Q^+(K_4) \end{bmatrix}. \quad \square$$

3. Conclusion

The MP inverse matrix of the non-normal cyclic subgroup graph has been obtained for the dihedral groups and generalized quaternion groups. The size of the MP inverse matrix of the digraph in this research has been determined and stated for every n odd and n even positive integer. As for the dihedral groups, the size of the MP inverse matrix is $(3n - 1) \times 2n$ when n is odd and $(3n - 2) \times 2n$ when n is even. Meanwhile, the size of the MP inverse matrix for generalized quaternion groups is $(14n - 2) \times 4n$ when n is odd and $(14n - 4) \times 4n$ when n is even. The MP inverse matrix is given in the matrix size and the entries are block matrices since the graph is disconnected and the incidence matrix is not unique depending on how the

vertices and edges are labelled. However, the *MP* inverse matrix is unique to the incidence matrix.

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