

HYBRID NANOFLUID FLOW OVER A CYLINDRICAL SURFACE WITH THE DOUBLE STRATIFICATION EFFECTS

(Aliran Nanobendalir Hibrid pada Permukaan Silinder dengan Kesan Penstrataan Berganda)

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ABSTRACT

The present problem studies hybrid nanofluid (HNF) flow along a cylindrical surface, incorporating the effects of double stratification and heat generation. The boundary layer model is reduced to a system of ordinary differential equations (ODEs) by introducing dimensionless variables, which is then computed numerically using the *bvp4c* routine in MATLAB. The Al_2O_3 -Cu/water HNF was used, combining two types of solid nanoparticles, namely, alumina (Al_2O_3) and copper (Cu), and water as the base fluid. For some physical parameters, it is discovered that dual solutions are feasible, where the critical points of the solution occur at the lowest suction strength ($S \leq 1$) and in the shrinking region ($\chi < 0$). By comparing the Al_2O_3 - H_2O nanofluid ($\phi_1 = 10\%, \phi_2 = 0\%$) with Al_2O_3 -Cu/ H_2O HNF ($\phi_1 = 10\%, \phi_2 = 1\%, 2\%$), increasing the volume fraction of hybrid nanoparticles enhances both dynamic viscosity and surface shear stress. Moreover, it is examined that skin friction and the heat and mass transfer rates at the surface are higher for a cylinder than for a flat plate. Finally, the results found that between the two solutions, the first solution maintains physical stability throughout temporal evolution.

Keywords: thermal stratification; solutal stratification; stagnation-point flow; heat generation; hybrid nanofluid; dual solutions; stability analysis

ABSTRAK

Kajian ini melibatkan aliran nanobendalir hibrid (HNF) di sepanjang permukaan silinder dengan mengambil kira kesan penstrataan berganda serta penjanaan haba. Model lapisan sempadan menakluk telah dijemakan kepada satu sistem persamaan pembezaan biasa (ODE) menggunakan set pemboleh ubah tidak bermatra, seterusnya kaedah berangka *bvp4c* di MATLAB digunakan untuk menyelesaikan sistem tersebut. HNF Al_2O_3 -Cu/air digunakan, yang menggabungkan dua jenis nanozarah pepejal iaitu alumina oksida (Al_2O_3) dan kuprum (Cu), dengan air sebagai bendalir asas. Bagi beberapa parameter fizikal, didapati bahawa penyelesaian dual wujud, di mana titik kritikal penyelesaian berlaku pada kekuatan sedutan terendah ($S \leq 1$) dan permukaan mengecut ($\chi < 0$). Dengan membandingkan nanobendalir Al_2O_3 -air ($\phi_1 = 10\%, \phi_2 = 0\%$) dengan HNF Al_2O_3 -Cu/air ($\phi_1 = 10\%, \phi_2 = 1\%, 2\%$), didapati bahawa pecahan isipadu nanozarah hibrid meningkatkan kelikatan dinamik dan tegasan ricih permukaan. Selain itu, diperhatikan bahawa geseran kulit serta kadar pemindahan haba dan jisim pada permukaan adalah lebih tinggi bagi silinder berbanding plat rata. Akhirnya, hasil kajian mendapati bahawa antara dua penyelesaian yang diperolehi, penyelesaian pertama mengekalkan kestabilan fizikal sepanjang evolusi masa.

Kata kunci: aliran titik genang; penstrataan terma; penstrataan kepekatan; penjanaan haba; nanobendalir hibrid; analisis kestabilan

1. Introduction

The study of heat and mass transfer is fundamentally important across numerous engineering and industrial fields, such as cooling by transpiration, oil recovery systems, thrust bearing development, radial diffuser configurations, and flow drag control (Attia 2010). Heat transfer is the flow of heat due to temperature differences and the resulting temperature distribution and changes. The phenomenon of mass transfer is encountered in numerous natural and industrial processes, such as phase-change operations, insulation technologies, food manufacturing, cooling tower performance, absorption processes, and the transport of nutrients in living tissues (Khan & Nadeem 2021). According to Malik *et al.* (2016), the significance of fluid dynamics and heat transfer over stretched surfaces during manufacturing processes such as copper wire drawing and glass blowing. The drawing of copper wire fundamentally differs from the extrusion process, as it relies on pulling the material through a die rather than applying pressure to push it through. Nevertheless, extrusion remains a key technique, particularly for manufacturing plastic and metal sheets. Careful monitoring of cooling and heat transfer during extrusion is critical, as these parameters strongly affect the properties and quality of the finished product. While water and air are traditionally employed as cooling media, recent studies advocate for alternative cooling agents that allow for a slower and more controlled solidification process. In addition, Yahaya *et al.* (2023) highlighted that employing the Cattaneo–Christov heat flux formulation provides deeper insight into heat and mass transfer mechanisms in non-Newtonian Sutterby fluids relevant to polymer processing. This approach enables a more accurate representation of diffusivity, thermal conductivity, and unsteady fluid effects.

Stratification refers to the development or layering of materials that arises from differences in thermal conditions, concentration levels, or density variations among interacting fluids. This phenomenon is significant in both natural and industrial contexts, such as thermal layering in oceans and reservoirs, groundwater systems, and energy storage processes (Hayat *et al.* 2017). According to Rehman *et al.* (2016), in reservoir systems, biological activity and energy storage effects contribute to oxygen depletion in the lower layers. The adverse impact of this phenomenon can be alleviated by employing thermal stratification, i.e., thermal stratification in lakes is characterized by a vertical temperature gradient within the water column, resulting from variations in water density induced by temperature differences. Many recent literatures also considered the stratification on the boundary layer flow (Hayat *et al.* 2016; Ibrahim & Makinde 2013; Mohd Sohut *et al.* 2017; Srinivasacharya & Upendar 2013). Additionally, Hayat *et al.* (2015) discussed that stratification plays a key role in governing heat and mass transfer mechanisms. In scenarios where heat and mass are transferred concurrently, evaluating the effects of double stratification on convective flow becomes particularly important. For example, Khashi'ie *et al.* (2019d) presented the analysis of a micropolar fluid with double stratification on a permeable stretching/shrinking sheet and found the dual solution for both buoyancy flows. In the same year, Khashi'ie *et al.* (2019a; 2019c) extend their research on double stratification in nanofluid flow. The effects of thermophoresis and Brownian motion were integrated using Buongiorno's nanofluid model in the analysis. Khan and Nadeem (2021) conducted an analysis on a rotating Maxwell fluid, emphasising the role of activation energy under dual stratification conditions. The flow analysis is made between an exponentially stretching and a linear sheet. In recent years, double stratification has gained considerable attention, resulting in continuous advancements and refinements, as shown in the studies of Yahaya *et al.* (2023) and Joyce and Kandasamy (2025).

Nanotechnology has swiftly evolved in recent years, and stability concerns and low thermal conductivity could be controlled by merging more than one nanoparticle element. Prior to that, hybrid nanofluid (HNF) had been receiving a lot of attention from researchers (Khan *et al.* 2020; Zainal *et al.* 2020; Ahmad *et al.* 2021; Ali *et al.* 2021; Ijaz Khan *et al.* 2021) and has been considered a topic due to its significance in enhancing the rate of heat transfer and stability. Also, it has a synergistic effect on nanomaterials and is attributed to a good aspect ratio. Devi and Devi (2016) mentioned that HNF is an advanced type of convectional heat transfer fluid, and due to its higher thermal conductivity, it is better than nanofluid. In addition, examples of its applications are mostly in heat transfer fields such as generator cooling, nuclear system cooling, engine cooling, and vehicle thermal management. Besides that, Khashi'ie *et al.* (2019b) reported that in hybrid nanofluids (HNF), thermal conductivity is a function of several interacting factors, including nanoparticle morphology, dimensional scale, volume fraction, and the nature of the base fluid. Both metallic nanoparticles (including Cu, Ni, Ag, and Au) and metal-oxide nanoparticles (such as Al_2O_3 , SiO_2 , CuO , and Fe_2O_3) are widely considered in related investigations. Further, Khan *et al.* (2021) studied heat transfer of hybrid nanoparticles (Ag-TiO₂/various base fluid) in micropolar fluid along with the different shape factors (i.e., sphere, cylinder, blade, lamina). Results revealed that both the lamina shape factor and a 50:50 water–ethylene glycol mixture contributed to an elevated Nusselt number. The impact of thermal stratification and mixed convection of a water-based copper-alumina HNF was analysed by Khashi'ie *et al.* (2019e). They observed that when the least suction parameter is applied on the stretching or shrinking sheet, heat transfer in the HNF is enhanced, but higher suction values lead to a reduction in this rate. The extended variants of the Xue (2005) model and the Yamada-Ota (1980) model were used by Abbas *et al.* (2021) to examine the HNF flow along a moving cylinder while an inclined magnetic field is applied. In all circumstances, velocity and temperature gradients, based on the comparative analysis indicate that the Yamada-Ota (1980) formulation predicts superior heat transfer performance relative to the Xue (2005) model. More recently, an upgraded version of the Yamada-Ota (1980) and Xue (2005) models incorporating a ternary hybrid nanofluid for thermal heat transport has been reported by Ishtiaq *et al.* (2025) to more accurately describe thermophysical characteristics.

Inspired by the previous literature, this work focuses on investigating the coupled heat and mass transfer characteristics of HNF along a cylindrical surface and to scrutinise the behaviors of HNF flow when double stratification and heat generation are considered. The next objective is to obtain a dual solution when the suction parameter is applied in the study and suggest a stability inquiry to prove the physical outcome. Present study is extended, following Mohd Sohit *et al.* (2017), Hayat *et al.* (2017), Khashi'ie *et al.* (2020) and Hafeez *et al.* (2021) by considering the HNF Al_2O_3 (alumina)-Cu (copper)/Water. Therefore, we are confident that our research is distinct and will have a significant impact on industrial and engineering purposes.

2. Problem Formulation

We examine a steady, incompressible mixed convection hybrid nanofluid (HNF) flow over a vertical cylinder of radius R with double stratification, as shown in Figure 1. The wall is considered impermeable, while the cylinder undergoes axial stretching or shrinking at a linearly varying velocity $u_w(x) = Bx/L$ and the free stream velocity $u_e(x) = Ax/L$, where L denotes the reference length of the cylinder, A is a positive constant and B is the reference velocity. The velocity components u and v correspond to the axial x and radial r directions,

respectively, while x and r denote the coordinates measured along the axial and radial axes of the cylinder. The symbol g represents the gravitational acceleration. The prescribed surface temperature, $T_w(x) = T_0 + bx/L$ and concentration, $C_w(x) = C_0 + dx/L$ at the cylinder wall are assumed to be higher than the corresponding ambient temperature, $T_\infty(x) = T_0 + cx/L$ and concentration, $C_\infty(x) = C_0 + ex/L$. Both the ambient temperature and concentration are considered to vary linearly with position, where T_0 and C_0 represent the reference temperature and concentration, respectively, and b, c, d , and e denote positive constants.

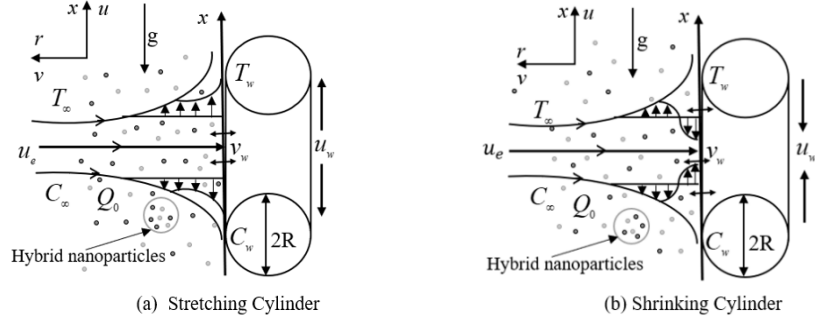


Figure 1: Physical model of a stretching and shrinking cylindrical surface

The mathematical description of the problem is established in terms of Cartesian coordinates (x, y) as presented in Refs. (Mohd Sohut *et al.* 2017; Hayat *et al.* 2017; Khashi'ie *et al.* 2020; Hafeez *et al.* 2021) is of the form:

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} - u_e \frac{\partial u_e}{\partial x} = \frac{\nu_{Hnf}}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{(\rho\beta_T)_{Hnf}}{\rho_{Hnf}} g(T - T_\infty) + \frac{(\rho\beta_C)_{Hnf}}{\rho_{Hnf}} g(C - C_\infty) \quad (2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial r} = \frac{k_{Hnf}}{(\rho C_p)_{Hnf}} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{Q_0}{(\rho C_p)_{Hnf}} (T - T_\infty) \quad (3)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial r} = D_B \left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} \right) \quad (4)$$

subject to

$$\begin{aligned} u(x, r) &= u_w(x), & v(x, r) &= v_w = -\frac{R}{r} \sqrt{\frac{Av_f}{L}} S, & T(x, r) &= T_w(x), \\ C(x, r) &= C_w(x) & & & & \text{at } r = R \\ u(x, r) &\rightarrow u_e(x), & T(x, r) &\rightarrow T_\infty, & C(x, r) &\rightarrow C_\infty & \text{as } r \rightarrow \infty \end{aligned} \quad (5)$$

Here, the fluid temperature and concentration are represented by T and C , respectively. The gravitational acceleration is given by g , β_T and β_C denote the thermal and solutal

expansion coefficients, Q_0 represents the heat generation or absorption, and D_b denotes the molecular diffusivity.

The volume fractions of alumina (Al_2O_3) and copper (Cu) nanoparticles are denoted ϕ_1 and ϕ_2 , with the subscripts $n1$ and $n2$ refer to their respective solid components. The subscripts f , nf and Hnf correspond to the base fluid, the nanofluid and HNF, respectively, as summarised in Table 1. Additionally, the physical properties of alumina, copper, and water (the base fluid) are listed in Table 2.

Table 1: Thermophysical correlations of nanofluid and HNF (Devi & Devi 2016; Waini *et al.* 2020a).

Properties	Nanofluid	Hybrid Nanofluid
Density, ρ	$\rho_{nf} = (1 - \phi_1)\rho_f + \phi_1\rho_{n1}$	$\rho_{Hnf} = (1 - \phi_2)\left[(1 - \phi_1)\rho_f + \phi_1\rho_{n1}\right] + \phi_2\rho_{n2}$
Dynamic viscosity, μ	$\mu_{nf} = \frac{\mu_f}{(1 - \phi_1)^{5/2}}$	$\mu_{Hnf} = \frac{\mu_f}{(1 - \phi_1)^{5/2}(1 - \phi_2)^{5/2}}$
Thermal conductivity, k	$\frac{k_{nf}}{k_f} = \frac{k_{n1} + 2k_f - 2\phi_1(k_f - k_{n1})}{k_{n1} + 2k_f + \phi_1(k_f - k_{n1})}$	$\frac{k_{Hnf}}{k_f} = \frac{k_{n2} + 2k_{nf} - 2\phi_2(k_{nf} - k_{n2})}{k_{n2} + 2k_{nf} + \phi_2(k_{nf} - k_{n2})}$ where $\frac{k_{nf}}{k_f} = \frac{k_{n1} + 2k_f - 2\phi_1(k_f - k_{n1})}{k_{n1} + 2k_f + \phi_1(k_f - k_{n1})}$
Heat capacity, (ρC_p)	$(\rho C_p)_{nf} = (1 - \phi_2)(\rho C_p)_f + \phi_1(\rho C_p)_{n1}$	$(\rho C_p)_{Hnf} = (1 - \phi_2)\left[(1 - \phi_1)(\rho C_p)_f + \phi_1(\rho C_p)_{n1}\right] + \phi_2(\rho C_p)_{n2}$

Table 2: Thermophysical properties (Oztop & Abu-Nada 2008)

Properties	Al_2O_3	Cu	Water
Prandtl number, Pr			6.2
Density, ρ	3970	8933	997.1
Thermal conductivity, k	40	400	0.613
Heat capacity, C_p	765	385	4179

To obtain the similarity solutions, the following transformation variables are employed, as proposed by Hayat *et al.* (2017), Khashi'ie *et al.* (2020), and Hafeez *et al.* (2021):

$$\Psi = \sqrt{\frac{Av_f x^2}{L}} Rf(\eta), \quad \eta = \frac{r^2 - R^2}{2R} \sqrt{\frac{A}{v_f L}}, \quad u = \frac{Ax}{L} f'(\eta),$$

$$v = -\frac{R}{r} \sqrt{\frac{Av_f}{L}} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_0}, \quad \phi(\eta) = \frac{C - C_\infty}{C_w - C_0}$$
(6)

Here, the notation $(')$ signifies the derivative taken with respect to η , $f(\eta)$ and $\theta(\eta)$ represent the stream function and temperature expressed in dimensionless form, respectively, and the similarity variable, η .

By applying the similarity transformations given in Eq. (6) to Eqs. (2) to (4), the following set of transformed ODEs are as follows:

$$\frac{\mu_{Hnf}}{\rho_{Hnf}} \frac{\mu_f}{\rho_f} [(2K\eta + 1)f''' + 2Kf''] + ff'' + 1 - f'^2 + \lambda_T \theta(\eta) + \lambda_C \phi(\eta) = 0 \quad (7)$$

$$\frac{1}{Pr} \frac{k_{Hnf}/k_f}{(\rho C_p)_{Hnf}/(\rho C_p)_f} [(1 + 2K\eta)\theta'' + 2K\theta'] - (\theta + \varepsilon_1)f' + f\theta' + \frac{1}{(\rho C_p)_{Hnf}/(\rho C_p)_f} \delta \theta = 0 \quad (8)$$

$$\frac{1}{Sc} [(1 + 2K\eta)\phi'' + 2K\phi'] - (\phi + \varepsilon_2)f' + f\phi' = 0 \quad (9)$$

The new boundary conditions are:

$$\begin{aligned} f(0) = S, \quad f'(0) = \chi, \quad \theta(0) = 1 - \varepsilon_1, \quad \phi(0) = 1 - \varepsilon_2, \quad \text{at } \eta = 0 \\ f'(\eta) \rightarrow 1, \quad \theta(\eta) \rightarrow 0, \quad \phi(\eta) \rightarrow 0, \quad \text{as } \eta \rightarrow \infty \end{aligned} \quad (10)$$

where λ_T , λ_C , K , δ , Gr_x , Gr_x^* , ε_1 , ε_2 , χ , Re_x , Pr , and Sc are the thermal buoyancy parameter, solutal buoyancy parameter, curvature parameter, heat generation parameter, local Grashof number due to temperature, local Grashof number due to concentration, thermal stratification parameter, solutal stratification parameter, constant stretching/shrinking parameter, local Reynolds number, Prandtl number and Schmidt's number, respectively. Here, $S < 0$ and $S > 0$ correspond to the parameters associated with fluid injection and suction, respectively.

$$\begin{aligned} \lambda_T = \frac{Gr_x}{Re_x^2}, \quad \lambda_C = \frac{Gr_x^*}{Re_x^2}, \quad K = \frac{1}{R} \sqrt{\frac{\nu_f L}{A}}, \quad \delta = \frac{LQ_0}{A(\rho C_p)_f}, \\ Gr_x = \frac{g(\beta_T)_{Hnf}(T_w - T_0)x^3}{\nu_f^2}, \quad Gr_x^* = \frac{g(\beta_C)_{Hnf}(C_w - C_0)x^3}{\nu_f^2}, \\ \varepsilon_1 = \frac{c}{b}, \quad \varepsilon_2 = \frac{e}{d}, \quad \chi = \frac{B}{A}, \quad Pr = \frac{\nu_f}{\alpha_f}, \quad Sc = \frac{\nu_f}{D_B} \end{aligned} \quad (11)$$

where $\lambda_T = 0$, $\lambda_T > 0$, and $\lambda_T < 0$ correspond to the thermal convective force, assisting and opposing flow, respectively. Meanwhile, $\lambda_C = 0$, $\lambda_C > 0$, and $\lambda_C < 0$ represent the solutal convective force, aiding flow and opposing flow, respectively. It is also important to note that $\chi > 0$ and $\chi < 0$ are correspond to the cylinder's stretching and shrinking.

The key physical quantities characterising this flow include the skin friction coefficient, C_f , the local Nusselt number, Nu_x and the local Sherwood number, Sh_x which are defined as follows:

$$C_f = \frac{\mu_{Hnf}}{\rho_f u_e^2(x)} \left(\frac{\partial u}{\partial r} \right)_{r=R}, \quad Nu_x = -\frac{x k_{Hnf}}{k_f (T_w - T_\infty)} \left(\frac{\partial T}{\partial r} \right)_{r=R},$$

$$Sh_x = -\frac{x}{(C_w - C_\infty)} \left(\frac{\partial C}{\partial r} \right)_{r=R}$$

Using (6) and (12), we get

$$\text{Re}_x^{\frac{1}{2}} C_f = \frac{\mu_{Hnf}}{\mu_f} f''(0), \quad \text{Re}_x^{-\frac{1}{2}} (1 - \varepsilon_1) Nu_x = -\frac{k_{Hnf}}{k_f} \theta'(0),$$

$$\text{Re}_x^{-\frac{1}{2}} (1 - \varepsilon_2) Sh_x = -\phi'(\eta)$$

3. Temporal Stability Analysis

For a specific set of physical parameters result in multiple solutions to Eqs. (7)–(10). Hence, temporal stability analysis is essential to ascertain the solutions that remain physically viable over time. This stability analysis approach was first introduced by Merkin (1986), with Weidman *et al.* (2006) extending it through the used of a dimensionless time variable, τ to access the temporal persistence of the solutions. The findings confirmed the physical stability of the first solution, while the second solution was found to be unstable. In this work, an unsteady flow problem is considered for the stability analysis, and the modified governing equations, Eqs. (2)-(4) are expressed as follows:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} - u_e \frac{\partial u_e}{\partial x} = \frac{\nu_{Hnf}}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{(\rho \beta_T)_{Hnf}}{\rho_{Hnf}} g (T - T_\infty)$$

$$+ \frac{(\rho \beta_C)_{Hnf}}{\rho_{Hnf}} g (C - C_\infty)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial r} = \frac{k_{Hnf}}{(\rho C_p)_{Hnf}} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{Q_0}{(\rho C_p)_{Hnf}} (T - T_\infty)$$

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial r} = D_B \left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} \right)$$

The variables reformulated in accordance with Eq. (6) are expressed as

$$\Psi = \sqrt{\frac{A v_f x^2}{L}} R f(\eta, \tau), \quad \eta = \frac{r^2 - R^2}{2R} \sqrt{\frac{A}{v_f L}}, \quad \tau = \left(\frac{A}{L} \right) t, \quad u = \frac{Ax}{L} \frac{\partial f}{\partial \eta}(\eta, \tau),$$

$$v = -\frac{R}{r} \sqrt{\frac{A v_f}{L}} f(\eta, \tau), \quad \theta(\eta, \tau) = \frac{T - T_\infty}{T_w - T_0}, \quad \phi(\eta, \tau) = \frac{C - C_\infty}{C_w - C_0}$$

Using (17), Eqs. (14)-(16) can be rewritten as

$$\frac{\mu_{Hnf}}{\rho_{Hnf}} \frac{\mu_f}{\rho_f} \left[(2\eta K + 1) \frac{\partial^3 f}{\partial \eta^3} + 2K \frac{\partial^2 f}{\partial \eta^2} \right] + 1 + f \frac{\partial^2 f}{\partial \eta^2} - \frac{\partial f}{\partial \eta} \frac{\partial f}{\partial \eta} + \lambda_\tau \theta + \lambda_c \phi - \frac{\partial^2 f}{\partial \eta \partial \tau} = 0 \quad (18)$$

$$\begin{aligned} & \frac{1}{Pr} \frac{k_{Hnf}/k_f}{(\rho C_p)_{Hnf}/(\rho C_p)_f} \left[(2K\eta + 1) \frac{\partial^2 \theta}{\partial \eta^2} + 2K \frac{\partial \theta}{\partial \eta} \right] + f \frac{\partial \theta}{\partial \eta} - \frac{\partial f}{\partial \eta} \theta - \frac{\partial f}{\partial \eta} \varepsilon_1 \\ & + \frac{1}{(\rho C_p)_{Hnf}/(\rho C_p)_f} \delta \theta - \frac{\partial \theta}{\partial \tau} = 0 \end{aligned} \quad (19)$$

$$\frac{1}{Sc} \left[(2\eta K + 1) \frac{\partial^2 \phi}{\partial \eta^2} + 2K \frac{\partial \phi}{\partial \eta} \right] - \frac{\partial f}{\partial \eta} \phi - \frac{\partial f}{\partial \eta} \varepsilon_2 + f \frac{\partial \phi}{\partial \eta} - \frac{\partial \phi}{\partial \tau} = 0 \quad (20)$$

The conditions are:

$$\begin{aligned} f(0, \tau) = S, \quad \frac{\partial f}{\partial \eta}(0, \tau) = \chi, \quad \theta(0, \tau) = 1 - \varepsilon_1, \quad \phi(0, \tau) = 1 - \varepsilon_2 \quad \text{at} \quad \eta = 0 \\ \frac{\partial f}{\partial \eta}(\eta, \tau) \rightarrow 1, \quad \theta(\eta, \tau) \rightarrow 0, \quad \phi(\eta, \tau) \rightarrow 0, \quad \text{as} \quad \eta \rightarrow \infty \end{aligned} \quad (21)$$

In order to analyse the stability behavior of the solutions, perturbation equations are introduced based on the framework developed by Weidman *et al.* (2006).

$$f(\eta, \tau) = f(\eta) + e^{-\gamma \tau} F(\eta), \quad H(\eta, \tau) = h(\eta) + e^{-\gamma \tau} H(\eta), \quad \theta(\eta, \tau) = \theta(\eta) + e^{-\gamma \tau} G(\eta) \quad (22)$$

Here, γ denotes an unknown eigenvalue parameter that governs or determines the stability behavior of the solutions, whereas $F(\eta)$, $G(\eta)$, and $H(\eta)$ are a small relative to $f_0(\eta)$, $\theta_0(\eta)$, and $h_0(\eta)$, respectively. τ is assumed as zero, indicating an initial decline or rise of the solution (22) as proposed by Weidman *et al.* (2006). By employing Eq. (22), Eqs. (18) to (21) become:

$$\begin{aligned} & \frac{\mu_{Hnf}}{\rho_{Hnf}} \frac{\mu_f}{\rho_f} \left[(2\eta K + 1) F_0'''(\eta) + 2K F_0''(\eta) \right] + f_0''(\eta) F_0(\eta) + f_0(\eta) F_0''(\eta) \\ & - 2f_0'(\eta) F_0'(\eta) + \lambda_\tau G_0(\eta) + \lambda_c H_0(\eta) + \gamma F_0'(\eta) = 0 \end{aligned} \quad (23)$$

$$\begin{aligned} & \frac{1}{Pr} \frac{k_{Hnf}/k_f}{(\rho C_p)_{Hnf}/(\rho C_p)_f} \left[(2\eta K + 1) G_0''(\eta) + 2K G_0'(\eta) \right] + \theta_0'(\eta) F_0(\eta) + f_0(\eta) G_0'(\eta) \\ & - (\theta_0(\eta) + \varepsilon_1) F_0'(\eta) + \gamma G_0(\eta) + \left(\frac{1}{(\rho C_p)_{Hnf}/(\rho C_p)_f} \delta - f_0'(\eta) \right) G_0(\eta) = 0 \end{aligned} \quad (24)$$

$$\begin{aligned} & \frac{1}{Sc} [(2\eta K + 1)H_0''(\eta) + 2KH_0'(\eta)] + \phi_0'(\eta)F_0(\eta) + f_0(\eta)H_0'(\eta) \\ & - (\phi_0(\eta) + \varepsilon_2)F_0'(\eta) - f_0'(\eta)H_0(\eta) + \gamma H_0(\eta) = 0 \end{aligned} \quad (25)$$

subject to

$$\begin{aligned} F_0(0) &= 0, & F_0'(0) &= 0, & G_0(0) &= 0, & H_0(0) &= 0, \\ F_0'(\infty) &\rightarrow 0, & G_0(\infty) &\rightarrow 0, & H_0(\infty) &\rightarrow 0 \end{aligned} \quad (26)$$

For the present problem, we relaxed the derivative $F'(\eta) \rightarrow 0$, as $\eta \rightarrow \infty$ in Eq. (26) and replaced with $F''(0) = 1$. According to Harris *et al.* (2009), $F'(\eta) \rightarrow 0$ is replaced by $F''(0) = 1$ is to obtain the smallest eigenvalue.

4. Results and Discussion

The boundary value problem defined by the transformed momentum, energy, and concentration equations (Eqs. 7–9) with Eq. (10) is solved via MATLAB's *bvp4c* solver, utilising a finite difference formulation derived from the three-stage Lobatto IIIa method. An initial approximation at the mesh points is required, and the solver adapts the step size h as needed or $\Delta\eta$ (i.e., the increment between successive points in the similarity variable η) to achieve the desired level of accuracy. Therefore, accurate solutions and numerical convergence depend critically on the careful selection of the initial guess and boundary layer η_∞ . The additional details on the *bvp4c* solver are available in Shampine *et al.* (2003). Table 3 presents the validation of the numerical procedure by the comparison of $f''(0)$ issued by Khashi'ie *et al.* (2020) for different values of stretching/shrinking parameter (χ) and curvature parameter ($K = 0$) when $\varphi_1 = 0, \varphi_2 = 0.1, Pr = 6.2$ and $Sc = \lambda_T = \lambda_C = \delta = S = \varepsilon_1 = \varepsilon_2 = 0$. It is found that the present results for the first solution (1st Sol.) and the second solution (2nd Sol.) demonstrate a good agreement with those obtained by Khashi'ie *et al.* (2020).

In addition, Table 4 demonstrates the impacts of certain parameters on the physical quantities of interest for $Al_2O_3 - Cu/H_2O$ when $\varphi_1 = 0.1, Sc = 0.6, Pr = 6.2, \lambda_T = -1$ and $\lambda_C = 1$ (fixed throughout the numerical computation). It is noticed that an increase in φ_2 , leads to higher values of the local skin friction $f''(0)$, local Nusselt number $-\theta'(0)$ and local Sherwood number $-\phi'(0)$. Similar trends are apparent in K, S and ε_1 . However, the opposite outcome is observed when δ and ε_2 increase. Meanwhile, an increase in χ causes the values of $f''(0)$ to drop and $-\theta'(0)$ and $-\phi'(0)$ to rise, indicating a transition from the shrinking to the stretching condition.

Table 3: Results comparison for $f''(0)$ with different curvature parameter (K) and stretching/shrinking (χ) when $\varphi_1 = 0, \varphi_2 = 0.1, \text{Pr} = 6.2$ and $\text{Sc} = \lambda_T = \lambda_C = S = \varepsilon_1 = \varepsilon_2 = 0$

χ	Khashi'ie <i>et al.</i> (2020)				Present			
	$K = 0$		$K = 0.4$		$K = 0$		$K = 0.4$	
	1 st Sol.	2 nd Sol.	1 st Sol.	2 nd Sol.	1 st Sol.	2 nd Sol.	1 st Sol.	2 nd Sol.
2.0	-2.217106	-	-2.396084	-	-2.217106	-	-2.396085	-
1.0	0	-	0	-	0	-	0	-
0.5	0.837940	-	0.935394	-	0.837940	-	0.935395	-
0.0	1.447977	-	1.653867	-	1.447977	-	1.653870	-
-0.5	1.757032	-	2.096118	-	1.757032	-	2.096130	-
-1.15	1.271347	0.137095	2.003737	-0.131198	1.271347	0.137095	2.003737	-0.131199
-1.2	1.095419	0.274479	1.939265	-0.101872	1.095419	0.274479	1.939265	-0.101872
-1.2465	0.686382	0.651157	1.865189	-0.052290	0.686384	0.651157	1.865190	-0.052290

Table 4: Values of $f''(0)$, $-\theta'(0)$ and $-\phi'(0)$ when $\text{Pr} = 6.2$, $\text{Sc} = 0.6$, $\lambda_T = -1$ and $\lambda_C = 1$.

φ_2	K	S	δ	χ	ε_1	ε_2	$Al_2O_3 - Cu/H_2O$ ($\varphi_1 = 0.1$)		
							$f''(0)$	$-\theta'(0)$	$-\phi'(0)$
0	0.1	0.5	0.1	-1	0.01	0.02	2.48176259	0.86885645	0.56155712
0.01	0	0.5	0.1	-1	0.01	0.02	2.49843424	0.86238857	0.52419339
							2.61966348	0.96179581	0.58171686
							2.73510014	1.05259657	0.63590896
							2.61966348	0.96179581	0.58171686
	0.1	0.5	0.1	-1	0.01	0.02	3.03506657	1.99557501	0.69694336
							2.64246972	1.18507693	0.58470931
							2.61966348	0.96179581	0.58171686
							2.59232105	0.70881574	0.57806340
		0.7	0.1	-1	0.01	0.02	2.61966348	0.96179581	0.58171686
							1.92490374	2.98138462	0.95415513
							0.22233720	4.09027991	1.21053972
							2.60578756	0.94847767	0.57946991
		0.5	0.1	-1	0.01	0.02	2.61966348	0.96179581	0.58171686
							2.63345156	0.97489209	0.58393976
							2.64420719	0.98003186	0.58953373
							2.63196208	0.97097160	0.58563167
							2.61966348	0.96179581	0.58171686
0.02	0.1	0.5	0.1	-1	0.01	0.02	2.76019001	1.04077552	0.60079083

The numerical results obtained in this study are depicted in Figures 2 through 14 for $f''(0)$, $-\theta'(0)$ and $-\phi'(0)$ against S for various values of the curvature parameter K , copper nanoparticle volume fraction φ_2 , heat generation δ , stretching/shrinking parameter χ , thermal stratification ε_1 and solutal stratification ε_2 . It is observed that within a specific range of S ($S \geq S_c$), the solutions of Eqs. (7)–(9) exhibit non-uniqueness, while the absence of similarity solutions occurs when $S < S_c$ as a result of boundary layer detachment from the surface. It is noted that S_c is also known as the critical value. According to Waini *et al.* (2020b), to obtain the solution for the shrinking surface, a greater suction value is required and it may generate dual solutions. However, the current study obtained the least suction value ($S \leq 1$), and possibly it is due to the double stratification process that has a significant effect on reducing the suction value.

Figures 2(a) to 2(c) depict the differences of $f''(0)$, $-\theta'(0)$ and $-\phi'(0)$ with S for several K values when $\varphi_2 = 0.01$, $\delta = 0.1$, $\chi = -1$ (shrinking cylinder), $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.02$. A similar trend appears in Figures 2(a) and 2(b), illustrating the transition of the curvature parameter from a flat ($K = 0$) to a cylindrical surface ($K = 0.1, 0.2$) elevates the values of the measured quantities ($f''(0)$ and $-\theta'(0)$) for the upper branch solution, while they decrease for the second (lower branch) solution, which is not a physical solution. The increase in K reduces the cylinder diameter, diminishing the fluid surface interaction.

Figures 2(a) and 2(b) show a similar pattern where the increase of the curvature parameter from the outer surface of the cylinder behaves like a flat surface ($K = 0$) to a cylindrical surface ($K = 0.1, 0.2$), will increase the value of both $f''(0)$ and $-\theta'(0)$ for the first solution. But decrease for the second solution (lower branch), which is not a physical solution. The cylinder diameter decreases with increasing K , causing a reduction in fluid contact along the cylinder surface. Consequently, the resistance to fluid motion diminishes, enhancing the fluid flow and increasing the rate of heat transfer. Figure 2(c) shows that the value of $-\phi'(0)$ for both first and second solutions rises with increasing K . A higher curvature parameter also weakens the fluid's viscous effects, thereby promoting mass transfer. Based on the numerical results, the critical values of S_c for $K = 0, 0.1$ and 0.2 are $S_{c1} = 0.2086$, $S_{c2} = 0.1154$ and $S_{c3} = 0.1010$, respectively. Hence, it can be concluded that a cylindrical surface exhibits greater skin friction and enhanced heat and mass transfer rates compared to a flat plate.

Next, the differences of $f''(0)$, $-\theta'(0)$ and $-\phi'(0)$ with S for some values of φ_2 when $K = 0.1$, $\delta = 0.1$, $\chi = -1$ (shrinking cylinder), $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.02$ are given in Figures 3(a) to 3(c). The bifurcation values for the case of Al_2O_3/H_2O nanofluid ($\varphi_1 = 10\%, \varphi_2 = 0\%$) is $S_{c1} = 0.1887$ and for the $Al_2O_3 - Cu/H_2O$ HNF ($\varphi_1 = 10\%, \varphi_2 = 1\%, 2\%$) are $S_{c2} = 0.1554$ and $S_{c3} = 0.1248$, respectively. When comparing these two types of fluids, it is evident that HNFs contain a higher volume fraction of nanoparticles. A greater nanoparticle concentration tends to delay boundary layer separation. As illustrated in Figures 3(a) and 3(c), the first solution for $f''(0)$ and $-\phi'(0)$ increases with increasing of φ_2 , whereas the second solution exhibits the opposite behavior. It is noteworthy that when $f''(0) < 0$, the solid surface imposes a drag force on the fluid, while $f''(0) > 0$ signifies that the drag force acts on the cylinder surface, as shown in Figure 3(a). Additionally, when $f''(0) = 0$, it implies a frictionless solid-fluid interaction, where both move at the same velocity. Also, it is proven that the rise in the concentration of nanoparticles will enhance the fluid's viscosity performance. Thus, increasing drag force and mass transfer. Figure 3(b) further reveals the presence of dual behaviors for both branches with increasing of φ_2 . The values of $-\theta'(0)$ for the first solution increase at $S \leq 0.8$ and decrease when $S > 0.8$. While the second solution decreases when $S \leq 0.4$ and then increases when $S > 0.4$. Augmented heat transfer in the $S \leq 0.8$ region signifies that the HNF provides more effective heat transfer characteristics than those of the conventional nanofluid. For the $S > 0.4$ region, the heat transfer rate decreases due to the imposition of the stronger suction strength (see Waini *et al.* (2020b)).

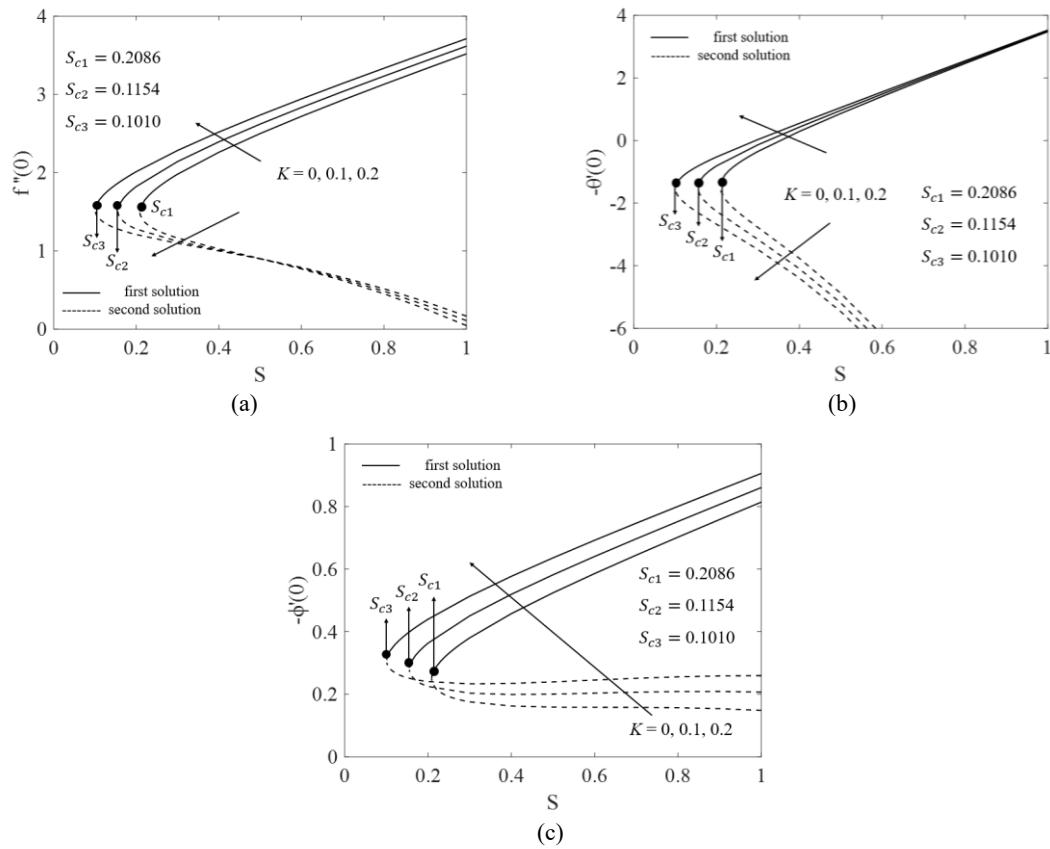


Figure 2: The effects of K against S when $\varphi_2 = 0.01$, $\delta = 0.1$, $\chi = -1$ (shrinking cylinder), $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.02$ for (a) $f''(0)$, (b) $-\theta'(0)$, and (c) $-\phi'(0)$

Furthermore, Figures 4(a) and 4(b) display the influence of heat generation δ on $f''(0)$ and $-\theta'(0)$, respectively, when $K = 0.1$, $\varphi_2 = 0.01$, $\chi = -1$ (shrinking cylinder), $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.02$. It is noted that the bifurcation values for $\delta = 0, 0.1$ and 0.2 are $S_{c1} = 0.0740$, $S_{c2} = 0.1554$ and $S_{c3} = 0.2283$, respectively. Figure 4(a) illustrates that the values of $f''(0)$ decrease for the first solution but increase for the second solution in the presence of heat generation ($\delta > 0$). A higher heat generation parameter will increase the fluid motions, make the friction between the nanoparticles faster, and exert a smaller drag force, which reduces the skin friction coefficient, as shown in the upper branch. However, the lower branch provides a contrary result, possibly due to the fact that it is not a physical solution. Figure 4(b) presents the values of $-\theta'(0)$ decrease for the first solution, in contrast to the presence of dual flow characteristics in the second solution where the values of $-\theta'(0)$ increase ($S \leq 0.5$) and then decrease ($S > 0.5$), respectively. Because the heat source's intensity is rather high, the value of $-\theta'(0)$ drops, suggesting that heat generation gives rise to a hot fluid layer near the wall instead of at the wall surface. As a consequence, heat transfer from the surface is reduced (Elbashbeshy 2012).

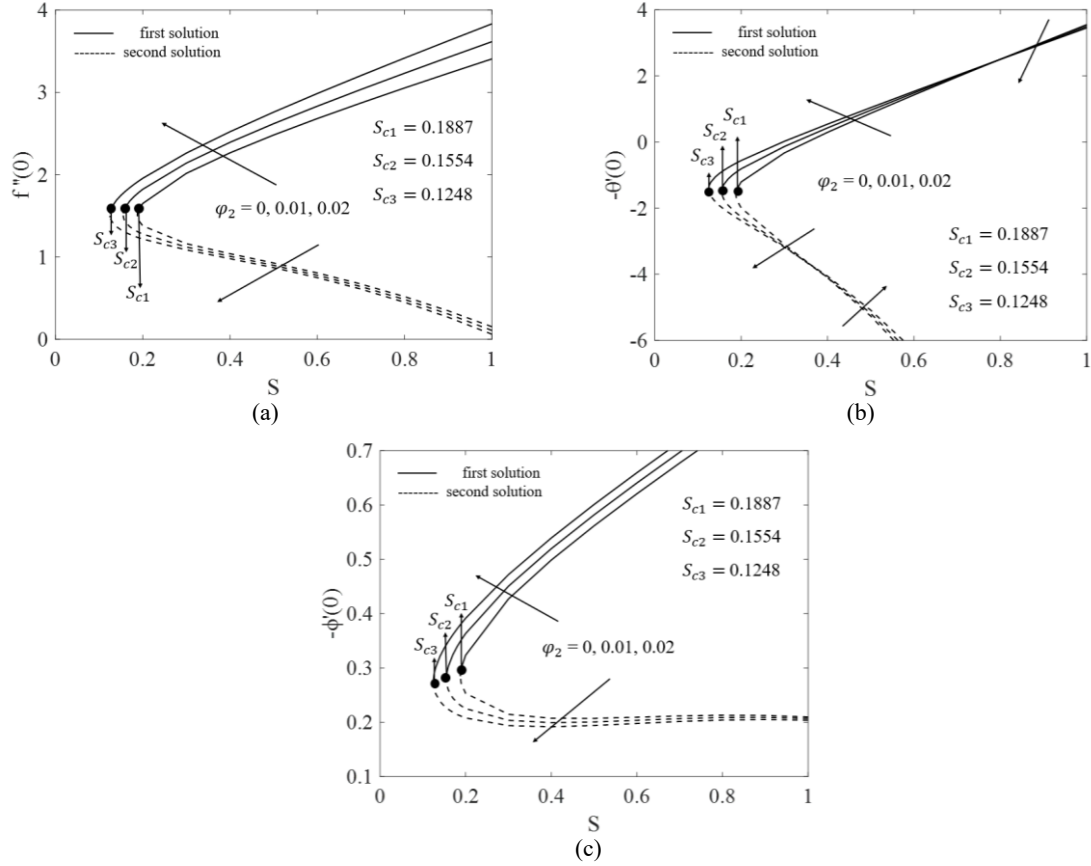


Figure 3: The effects of φ_2 against S when $K = 0.1$, $\delta = 0.1$, $\chi = -1$ (shrinking cylinder), $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.02$ for (a) $f''(0)$, (b) $-\theta'(0)$, and (c) $-\phi'(0)$

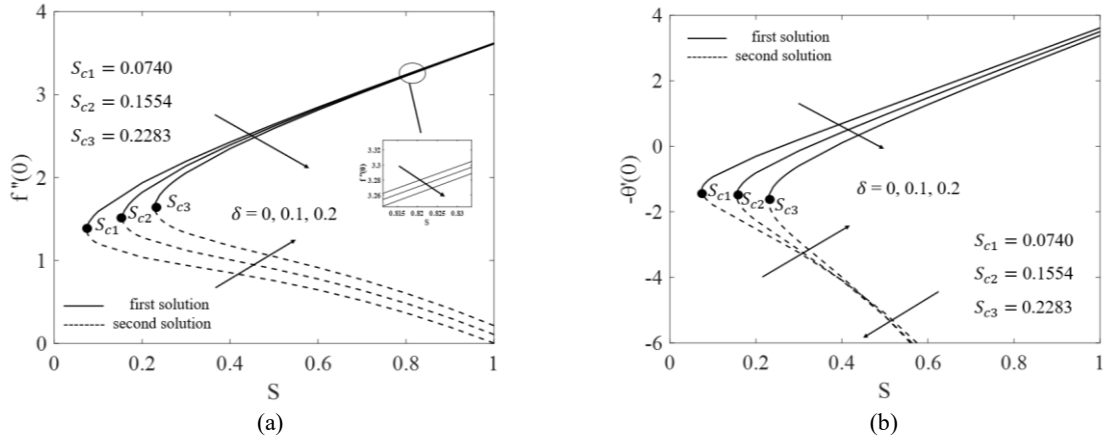


Figure 4: The effects of δ against S when $K = 0.1$, $\varphi_2 = 0.01$, $\chi = -1$ (shrinking cylinder), $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.02$ for (a) $f''(0)$, and (b) $-\theta'(0)$

The influences of the thermal stratification ε_1 on the values of $f''(0)$, $-\theta'(0)$ and $-\phi'(0)$, respectively, against S when $K = 0.1$, $\varphi_2 = 0.01$, $\delta = 0.1$, $\chi = -1$ (shrinking cylinder) and

$\varepsilon_2 = 0.02$ are demonstrated in Figures 5(a) to 5(c). It is discovered that the critical values of S are $S_{c1} = 0.1624$, $S_{c2} = 0.1554$ and $S_{c3} = 0.1479$ for $\varepsilon_1 = 0, 0.01$ and 0.02 , respectively. Physically, thermal stratification ε_1 refers to the phenomenon in which a fluid body forms distinct horizontal layers based on temperature differences. In still bodies of water, such as ponds and lakes, this results in the separation of layers with noticeably different temperatures. It is observed that the physical quantities of interest in this flow exhibit similar trend patterns in both solution branches. The effects of ε_1 increase the skin friction coefficient is illustrated in Figure 5(a). Elevated thermal stratification suppresses convective transport between the heated cylinder and the surrounding medium, intensifying viscous effects and consequently reducing the flow velocity. The rise of ε_1 also enhances the heat and mass transfer rates for the first solution, whereas the second solution has opposite characteristics, as shown in Figures 5(b) and 5(c).

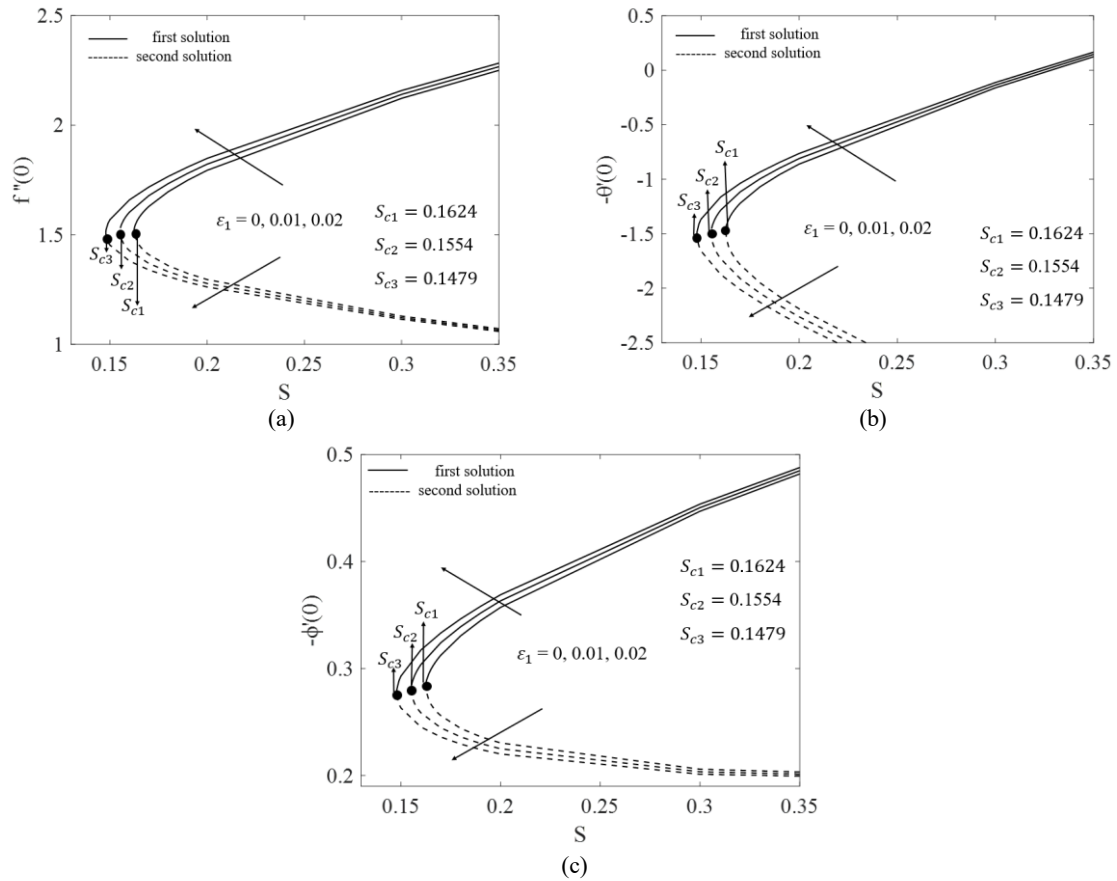


Figure 5: The effects of ε_1 against S when $K = 0.1$, $\delta = 0.1$, $\chi = -1$ (shrinking cylinder), $\varphi_2 = 0.01$ and $\varepsilon_2 = 0.02$ for (a) $f''(0)$, (b) $-\theta'(0)$, and (c) $-\phi'(0)$

Figures 6(a) and 6(b) portray the variations of $f''(0)$ and $-\phi'(0)$ with S for some values of ε_2 when $K = 0.1$, $\varphi_2 = 0.01$, $\delta = 0.1$, $\chi = -1$ (shrinking cylinder) and $\varepsilon_1 = 0.01$. As the solutal stratification ε_2 increases, both figures demonstrate contrasting profile patterns. The

values of $f''(0)$ and $-\phi'(0)$ for the upper branch decreases, but increases for the lower branch. Stronger solutal stratification attenuates the concentration gradient and associated buoyancy force, resulting in lower $f''(0)$ values in the upper branch. Additionally, the value of $-\phi'(0)$ declines for the increment of ε_2 for the first solution. This phenomenon results from the weakened convective interaction between the cylinder surface and the ambient concentration, indicating that an increase in ε_2 reduces both the rate of mass transfer and the boundary layer thickness. Moreover, the critical values for $\varepsilon_2 = 0, 0.01$ and 0.02 are $S_{c1} = 0.1437$, $S_{c2} = 0.1494$ and $S_{c3} = 0.1554$, respectively. It is discovered that the critical value S_c increases with the increment value of the solutal stratification ε_2 .

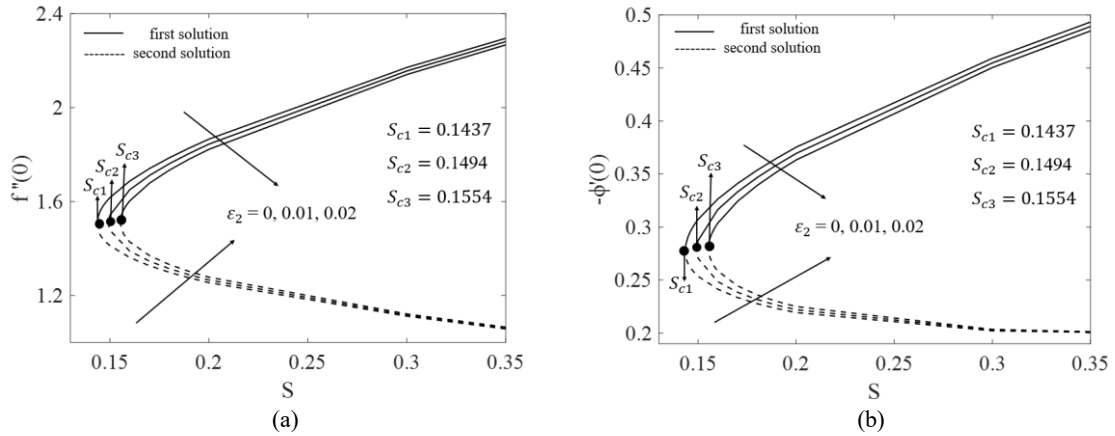


Figure 6: The effects of ε_2 against S when $K = 0.1$, $\varphi_2 = 0.01$, $\delta = 0.1$, $\chi = -1$ (shrinking cylinder), and $\varepsilon_1 = 0.01$ for (a) $f''(0)$ and (b) $-\phi'(0)$

Furthermore, Figures 7(a) to 7(c) illustrate the effects of the curvature parameter K in conjunction with stretching/shrinking parameter χ when $S = 0.5$, $\varphi_2 = 0.01$, $\delta = 0.1$, $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.02$, respectively. These figures indicate that dual solutions are possible for stretching ($\chi > 1$) and shrinking ($\chi < 1$) cylinder surfaces. The unique solution $\chi = \chi_c$ is obtained in the shrinking region up to a certain point, while dual solutions appear within a specific range of χ greater than a critical value χ_c . However, no solution is found for χ values smaller than χ_c . Beyond this critical limit $\chi < \chi_c$, the boundary layer detaches from the surface at $\chi = \chi_c$, making it impossible to obtain the solution. The minimum values of χ for which the solution exist are found to be $\chi_{c1} = -1.2825$, $\chi_{c2} = -1.3191$ and $\chi_{c3} = -1.3536$ for $K = 0, 0.1$ and 0.2 , respectively.

Figures 7(a) to 7(c) reveal that the cylindrical surface ($K = 0.1, 0.2$) increases the validity range of similarity solutions for Eqs. (7) to (9), thereby delaying boundary layer separation. This observation is consistent with the findings of Khashi'ie *et al.* (2020). The figures also exhibit a similar trend to that shown in Figures 2(a)-2(c) with respect to S . As illustrated in Figure 7(a), the skin friction coefficient for the first solution is higher for a cylindrical surface compared to a flat plate. With an increase in the curvature parameter, the cylinder radius diminishes, thereby restricting the region over which the fluid interacts with the surface. Consequently, the reduced flow resistance enhances fluid motion. Figure 7(b) reveals that as

K increases, the value of $-\theta'(0)$ for the first solution also increases. This indicates that higher curvature leads to greater fluid velocity, which raises the kinetic energy, resulting in an increase in temperature and, consequently, elevated heat transfer. In contrast, Figures 7(a) and 7(b) demonstrate an opposite trend for the second solution. Figure 7(c) further demonstrates that, with increasing K , the value of $-\phi'(0)$ rises for both the first and second solutions. These results are also consistent with the results shown in Figures 2(a)-2(c).

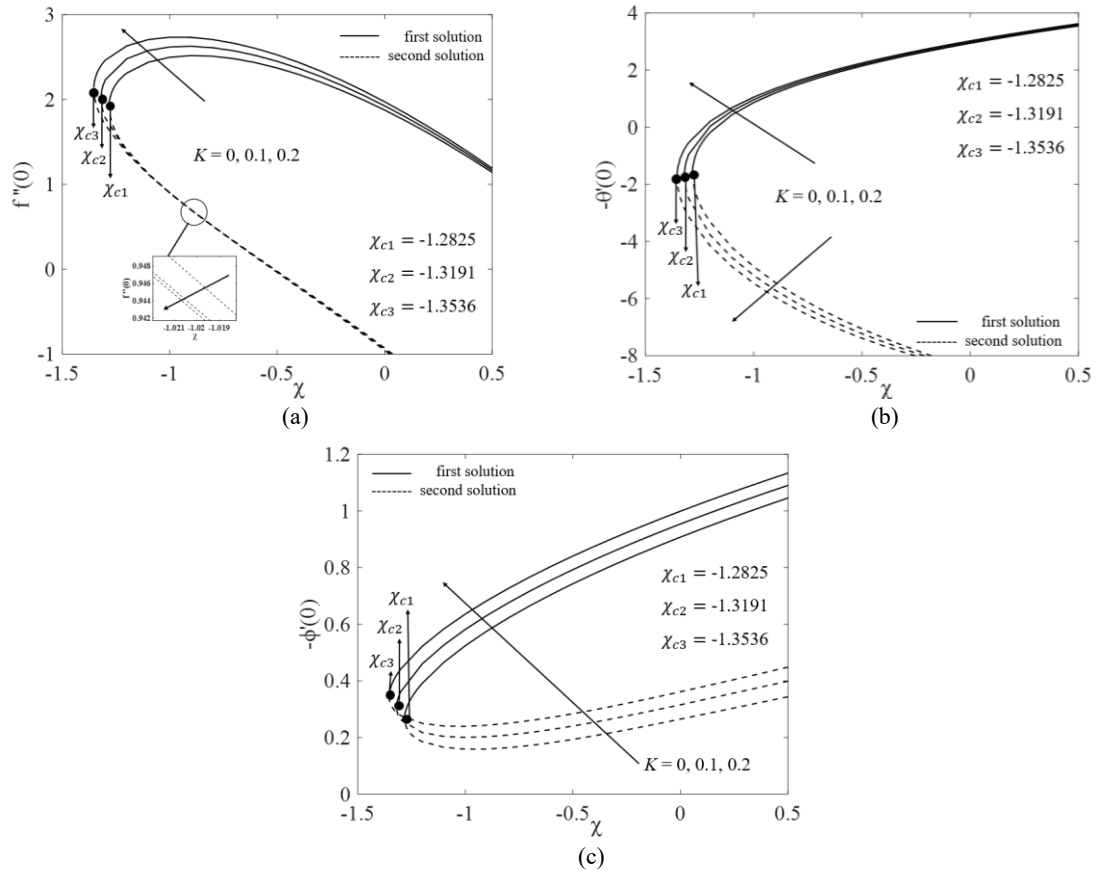


Figure 7: The effects of K against χ when $S = 0.5$, $\varphi_2 = 0.01$, $\delta = 0.1$, $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.02$ for (a) $f''(0)$, (b) $-\theta'(0)$, and (c) $-\phi'(0)$

Figures 8 and Figures 9 present the influence of selected non-dimensional parameters (i.e., S and φ_2) on the velocity $f'(\eta)$, temperature $\theta(\eta)$ and concentration $\phi(\eta)$ distributions for shrinking case ($\chi < 1$). The effects of the several values of S when $K = 0.1$, $\varphi_2 = 0.01$, $\delta = 0.1$, $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.02$ are shown in Figures 8(a) to 8(c). As S increases, the first solution exhibits higher velocity profile along with lower temperature and concentration profiles, whereas the second solution demonstrates the opposite trend, decreasing velocity profile with increasing temperature and concentration profiles. Figure 8(a) shows that the velocity profile increases with the increasing value of S . This suggests that adequate suction strength is required to obtain a solution for the shrinking case. The value of the shear stress profile for the first solution initially increases with increasing S , but for larger η , the profile declines, whereas the second solution exhibits the opposite trend. At the same time, stronger

suction causes a reduction in the temperature profile for the first solution as portrayed in Figure 8(b). This is significant with the shrinking case, which affects a reduction in heat transfer rate as well. This finding is supported by the discovery of prior research by Waini *et al.* (2020b). In addition, Figure 8(c) demonstrates the influence of S on the concentration profile. This profile shows the same profile pattern as the temperature profile. Concentration profile and mass boundary layer thickness decrease with stronger suction strength or a higher value of S .

Subsequently, the influences on the several values of φ_2 when $K = 0.1$, $S = 2.3$, $\delta = 0.1$, $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.02$ are shown in Figures 9(a) and 9(b). It is examined that increasing φ_2 leads to growth in the $f'(\eta)$ and $\theta(\eta)$ profiles for the upper branch, but decrease in the lower branch. Figure 9(a) demonstrates that an increase in φ_2 leads to higher nanoparticles concentration and a corresponding increase in the first solution. It will result in decreasing boundary layer thickness and delaying flow separation at the wall. Thus, this augments the fluid motion. These outcomes are in line with that illustrated in Figure 3(a). Furthermore, Figure 9(b) illustrates that the first branch increases as φ_2 rises, whereas the second solution exhibits a decreasing trend. The high concentration of φ_2 can also enhance energy transfer and elevate the fluid temperature, thereby enhancing the the boundary layer thickness. This result is also similar to that depicted in Figure 3(b) for $S \leq 0.8$ (least suction strength).

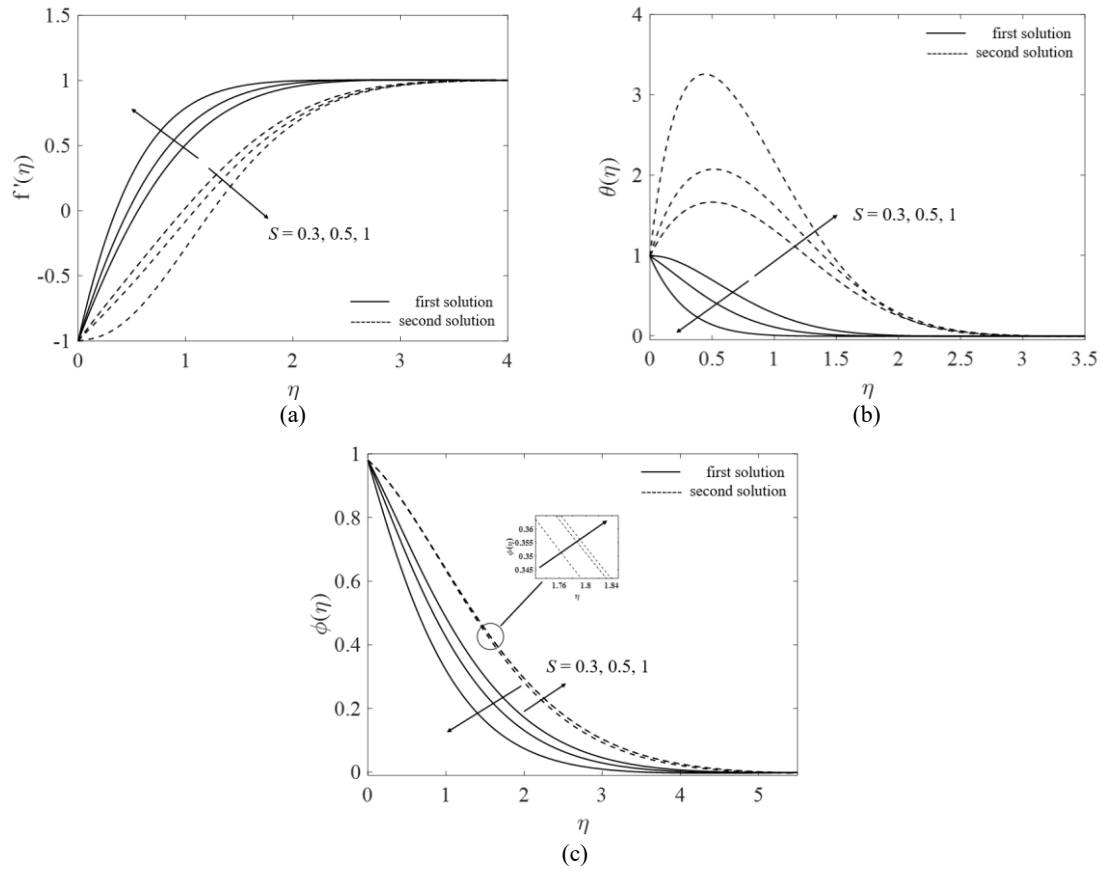


Figure 8: The effects of S when $K = 0.1$, $\varphi_2 = 0.01$, $\delta = 0.1$, $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.02$ for (a) $f'(\eta)$, (b) $\theta(\eta)$, and (c) $\phi(\eta)$

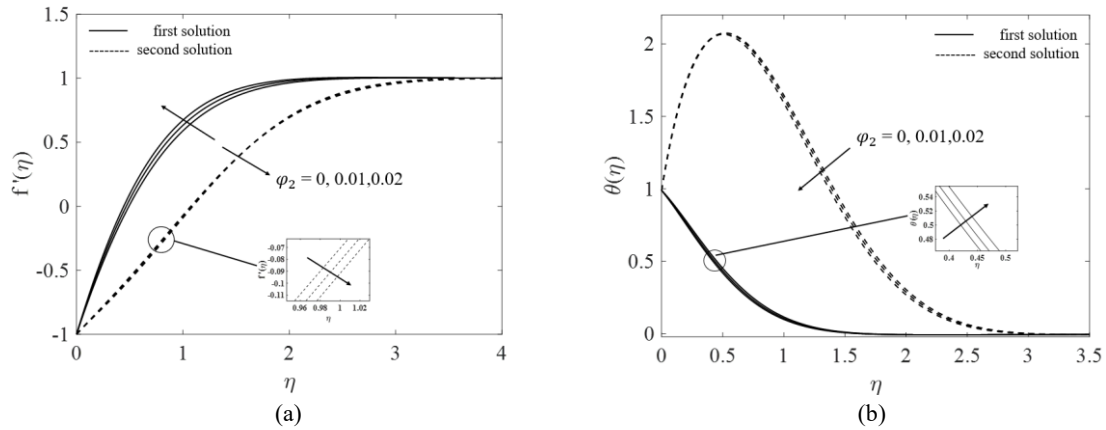


Figure 9: The effects of ϕ_2 when $K = 0.1$, $S = 2.3$, $\delta = 0.1$, $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.02$ for (a) $f'(\eta)$ and (b) $\theta(\eta)$

The variations of the minimum eigenvalues γ are depicted in Figure 10 over the suction parameter S when $K = 0.1$, $\phi_2 = 0.01$, $\delta = 0.1$, $\chi = -1$, $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.02$. From the illustration, the positive γ denotes that the first solution is stable, while the negative γ shows the contrary result. In accordance with Eq. (22), the flow remains stable when the initial disturbance decays over time, which occurs for $\gamma > 0$ (positive γ), i.e., $e^{-\gamma\tau} \rightarrow 0$ as $\tau \rightarrow \infty$. Meanwhile, the flow is unstable because of the small disturbances are amplified progressively over time, when $\gamma < 0$ (negative γ), i.e., $e^{-\gamma\tau} \rightarrow \infty$ as $\tau \rightarrow \infty$. It is also noticed that, as $S \rightarrow S_c = 0.1554$, $\gamma \rightarrow 0$, this implies that at the turning points, the flow transitions from the first to the second solution, where the values of S shift from positive to negative, indicating a change from stability to instability. This analysis reveals that the first solution remains stable and physically meaningful over time, whereas the second solution is unstable.

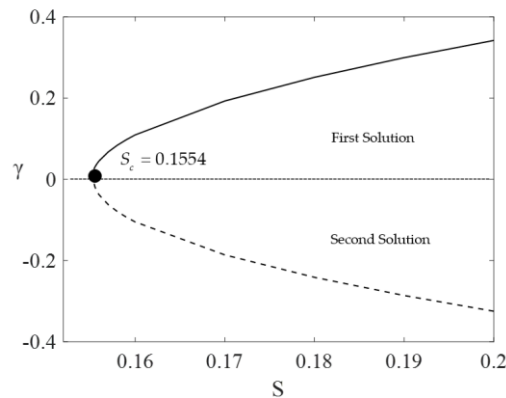


Figure 10: The smallest eigenvalues γ over the suction parameter S

5. Conclusion

This study considers the HNF along a cylindrical surface subjected to double stratification and heat generation. Those results are also validated and well compared with the previously published research. The variations in non-dimensional parameters affect the flow and the associated heat and mass transfer phenomena have been investigated. The findings are summarised below:

- The least suction strength ($S \leq 1$) is obtained, and possibly it is due to the double stratification process that has a significant effect on reducing the suction value.
- The cylinder exhibits higher skin friction and enhanced heat and mass transfer rates at its surface, compared to a flat plate.
- The rise in the concentration of nanoparticles will enhance the fluid's viscosity performance and thus accelerate heat transfer on the surface.
- A higher heat generation parameter will increase the fluid motions, make the friction between the nanoparticles faster, and exert a smaller drag force, which reduces the skin friction coefficient, the rate of heat transfer from the surface is reduced.
- The raises of solutal stratification ε_2 , results in a decline of $f''(0)$ and $-\phi'(0)$ for the upper branch, while the lower branch shows the opposite trend.
- The rise in ε_1 leads to higher values of $f''(0)$, $-\theta'(0)$ and $-\phi'(0)$ for the first solution, while the second solution demonstrates a reverse trend.
- This scrutiny reveals that the upper branch solution is confirmed to be physically valid and temporally stable, whereas the lower branch solution is not.

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