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ODD-EVEN CLIQUE EQUALITY IN THE COPRIME GRAPH OF THE GENERALISED QUATERNION GROUP

(*Persamaan Klik Ganjil-Genap pada Graf Ko-Perdana Kumpulan Kuaternion Teritlak*)

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ABSTRACT

Properties of various groups are investigated by associating them with graphs. The quaternion group, which is non-abelian, is one of the most explored groups. Several studies have been carried out on this group by associating it with various graphs, such as commuting graph, centre graph, conjugate graph and coprime graph. Meanwhile, the clique polynomial is determined to investigate the properties and relations of the cliques of graphs. The objectives of this paper are to determine and investigate the general forms of the clique polynomial of the coprime graph of the generalised quaternion group by using the prime factorisations of the order of the group. The algorithm for the coefficients of the clique polynomials is discussed using combinatorial methods and the partitioning of the sets of vertices. The results show that the total number of odd-sized cliques of the coprime graph of the generalised quaternion group is equal to the total number of even-sized cliques.

Keywords: clique; coprime graph; generalised quaternion group

ABSTRAK

Ciri-ciri pelbagai kumpulan dikaji dengan mengaitkannya kepada graf. Kumpulan kuaternion, yang merupakan kumpulan tak komutatif, merupakan salah satu kumpulan yang banyak diterokai. Pelbagai penyelidikan telah dijalankan terhadap kumpulan ini dengan mengaitkannya kepada pelbagai jenis graf seperti graf komutatif, graf pusat, graf konjugat dan graf ko-perdana. Sementara itu, polinomial klik ditentukan untuk mengkaji ciri-ciri dan hubungan antara klik dalam graf. Objektif kajian ini adalah untuk menentukan dan mengkaji bentuk umum polinomial klik bagi graf ko-perdana kumpulan kuaternion teritlak dengan menggunakan pemfaktoran perdana bagi tertib kumpulan tersebut. Algoritma bagi pekali polinomial klik dibincangkan dengan menggunakan kaedah kombinatorik dan pembahagian set bucu. Hasil kajian menunjukkan bahawa jumlah bilangan klik bersaiz ganjil adalah sama dengan jumlah bilangan klik bersaiz genap dalam graf ko-perdana kumpulan kuaternion teritlak.

Kata kunci: klik; graf ko-perdana; kumpulan kuaternion teritlak

1. Introduction

Graph theory is a mathematical concept applied to analyse the theoretical properties of abstract mathematical structures such as groups or rings through graph representations (Gross & Tucker 2001). Alaeiyan and Mohammadian (2011) mentioned that among these properties is the clique polynomial, which gives the number of cliques of different sizes in a graph. Previous works often associate various graphs with a specific group to represent different relations between the elements of the group and to investigate their properties.

The coprime graph of a group was defined by Ma *et al.* (2014), where the vertices correspond to each element of the group and edges are formed between elements whose orders are coprime. This graph has emerged as a new area for exploring more group properties or characteristics, particularly the generalised quaternion group, denoted as Q_{4n}, where n is a natural number.

The generalised quaternion group Q_{4n}, which is non-abelian for $n \geq 2$, has rich structural properties and is particularly amenable to a variety of combinatorial investigations. Recent works by Zahidah *et al.* (2021) include the investigation of some topological indices for the coprime graph of Q_{4n}. The graph invariants such as the chromatic number and clique number of the coprime graph of the generalised quaternion group are determined by Gayatri *et al.* (2023) and Nurhabibah *et al.* (2021), respectively. Munandar (2023) studied the structural properties of the coprime graph associated with the generalised quaternion group to gain deeper insights into the graph.

Building on the work of Najmuddin (2021), which explores graph polynomials associated with certain finite non-abelian groups, this paper examines the clique polynomial of the coprime graph of the generalised quaternion group. Throughout this paper, the Euler totient function is used to determine the number of coprime elements in Q_{4n} based on the orders of its elements. As noted by Lehmer (1932), this function provides an efficient way to account for the frequent occurrence of prime factorisations when determining the number of cliques in the graph. This idea is also inspired by the innovation of using pure combinatorial methods to determine graphical properties, as mentioned by Knill (2021).

While much is known about the calculation of cliques using the clique polynomial from various graphs, a general understanding of clique parity remains relatively unexplored. In this paper, we employ combinatorial methods and analysis to establish a novel result: a parity balance between the total number of odd-sized and even-sized cliques in the coprime graph of the generalised quaternion group. The significance of this work lies in enabling a more efficient calculation through the use of Euler's totient function, while the observed balance between the number of odd-sized and even-sized cliques may reflect the symmetric properties of the coprime graph. The following is our main result on the parity of odd- and even-sized cliques in the coprime graph of the generalised quaternion group. The proof will be presented in Section 4.

Theorem 1.1. *Let $\Gamma^{\mathrm{cp}}_{Q_{4n}}$ be the coprime graph of the generalised quaternion group Q_{4n}, and let ω be its clique number. For $\omega > 1$, the total number of odd-sized cliques equals the total number of even-sized cliques. Specifically, the sums of the counts of cliques of even size and odd size satisfy the following parity balance:*

$$\sum_{j=0}^{\lfloor \frac{\omega}{2} \rfloor} c_{2j} = \sum_{j=0}^{\lfloor \frac{\omega-1}{2} \rfloor} c_{2j+1}, \tag{1}$$

where $\lfloor \cdot \rfloor$ is the floor function and c_k is the number of size-k cliques.

2. Preliminaries

This section introduces definitions along with established statements from the literature that will be used in our analysis.

2.1. *Group Theory*

A group is a nonempty set G equipped with a binary operation $*$ that satisfies the following four properties: closure, associativity, the existence of an identity element, and the existence of inverses for every element. Among the many types of groups, one particularly well-behaved and fundamental class is that of cyclic groups, namely groups that can be generated by a single element.

Definition 2.1 (Rotman 2015)**.** *Cyclic group.* Let G be a group. For an element $a \in G$, the cyclic subgroup of G generated by a is denoted by $\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$, which consists of all powers of a, and $|\langle a \rangle| = n$.

If a generates the group G such that $G = \langle a \rangle$, then G is called a cyclic group generated by a. In other words, if $|G| = |\langle a \rangle| = n$, then G can also be stated as a cyclic group of order n, where a is known as the generator of G.

With this definition in place, we now recall a key result regarding the structure and generators of finite cyclic groups.

Theorem 2.2 (Rotman 2015)**.** *Let G be a cyclic group of order n, generated by an element $a \in G$. Then*

(1) $G = \langle a \rangle$ and $|G| = n$. Any element $a^k \in G$ also generates G if and only if $\gcd(k, n) = 1$.
(2) Let $\mathrm{gen}(G)$ be a set consisting of all the generators of G. Then $|\mathrm{gen}(G)| = \phi(n)$, where $\phi(n)$ is Euler's totient function, namely the total number of positive integers up to n that are relatively prime to n.

This brings us naturally to a closer examination of Euler's totient function, which plays a central role in both number theory and group theory.

Theorem 2.3 (Rosen 2019)**.** *The properties of Euler's totient, or phi, function $\phi(n)$ are as follows:*

(1) Let p be a prime number and k be a positive integer. Then

$$\phi\left(p^k\right) = p^k - p^{k-1} = p^k\left(1 - \frac{1}{p}\right).$$

(2) Let p_1 and p_2 be two relatively prime positive integers. Then

$$\phi\left(p_1 p_2\right) = \phi\left(p_1\right)\phi\left(p_2\right).$$

(3) Let $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ be a positive integer for $k_j \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$. Then

$$\phi(n) = n \prod_{i=1}^{m}\left(1 - \frac{1}{p_i}\right).$$

Before proceeding further, we also recall Lagrange's theorem, which provides a fundamental relationship between the orders of subgroups and the group itself, a result that supports many of the structural observations made above.

Theorem 2.4 (Gallian 2020)**.** Lagrange's theorem. *Let G be a finite group and H be a subgroup of G. Then $|H|$ is a divisor of $|G|$, or equivalently, $|H|$ divides $|G|$. In addition, the cardinality of the left or right cosets of H in G is equal to $\frac{|G|}{|H|}$, denoted by $|G : H|$, where $G : H$ is known as the index of H in G.*

2.2. *Graph Theory*

Any graph Γ can be written as a pair, that is $\Gamma = (V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is the nonempty set of vertices and $E(\Gamma)$ is the set of edges formed by unordered pairs of vertices from $V(\Gamma)$. With this foundational representation in mind, we now proceed to define some important types of graphs that will play a central role in our discussion, particularly those related to clique structures.

Definition 2.5 (Balakrishnan & Ranganathan 2012)**.** *Complete graph.* Let Γ be a graph. If every pair of distinct vertices of Γ forms an edge in Γ, then Γ is said to be a complete graph and can be denoted by K_n, where $n \in \mathbb{N}$ represents the total number of vertices in Γ.

Building upon the concept of a complete graph, we introduce the notion of a clique, which is essentially a complete subgraph embedded within a potentially larger incomplete graph.

Definition 2.6 (Balakrishnan & Ranganathan 2012)**.** *Clique.* A clique of a graph Γ is a subgraph of Γ that is complete. It can also be interpreted that a clique can be denoted as a complete graph.

The terms "clique of size-k" or "size-k clique" will be frequently used in this paper, where $k \in \mathbb{N}$. In addition, a clique of size-k in Γ is called a maximal clique if there is no other larger-sized clique in Γ that contains it.

To construct more complex graphs from simpler ones, such as cliques or independent sets, standard operations such as the join and the union of graphs are used.

Definition 2.7 (Balakrishnan & Ranganathan 2012)**.** *Join of two graphs.* Let Γ_1 and Γ_2 be two vertex-disjoint graphs. Then the join, $\Gamma_1 \vee \Gamma_2$, is a composition of graphs Γ_1 and Γ_2 where every vertex of Γ_1 is adjacent to every vertex of Γ_2.

Definition 2.8 (Balakrishnan & Ranganathan 2012)**.** *Union of two graphs.* Let Γ_1 and Γ_2 be two vertex-disjoint graphs. Then the union, $\Gamma_1 \cup \Gamma_2$, is a graph with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and edge set $E(\Gamma_1) \cup E(\Gamma_2)$.

Using these operations, we can describe a large class of structured graphs known as multipartite graphs, which include complete bipartite and complete multipartite graphs as special cases.

Definition 2.9 (Balakrishnan & Ranganathan 2012)**.** *Multipartite graph.* Let Γ be a graph. If its vertex set can be partitioned into k nonempty, vertex-disjoint and edgeless subsets, called partite sets, then Γ is called a multipartite graph or a k-partite graph.

According to Bazin and Bertaux (2018), each partite set in a multipartite graph can also be referred to as an independent set. Moreover, if every vertex in an independent set is adjacent to all vertices in the other independent sets within the graph Γ, then Γ is called a complete multipartite graph. Such a graph can be denoted by $K_{n_1,n_2,\ldots,n_\ell}$, where $\ell \in \mathbb{N}$ represents the number of partite sets, and each n_i denotes the size of the corresponding independent set.

Alternatively, this graph can be expressed as the join of its independent sets:

$$E_{n_1} \vee E_{n_2} \vee \cdots \vee E_{n_\ell},$$

where each E_{n_i} denotes an independent set consisting of $n_i \in \mathbb{N}$ vertices. Now that we have introduced key structural concepts in graph theory, including cliques and complete multipartite graphs, we proceed to define the clique polynomial, which serves as a powerful algebraic tool for encoding information about cliques of various sizes in a graph.

Definition 2.10 (Hoede & Li 1994)**.** *Clique polynomial.* The clique polynomial of a graph Γ is a polynomial in powers of x, denoted by

$$C(\Gamma; x) = \sum_{k=0}^{\omega(\Gamma)} c_k x^k,$$

where c_k represents the number of cliques of size k for $k = 0, 1, \ldots, \omega(\Gamma)$, and $\omega(\Gamma) \in \mathbb{N} \cup \{0\}$ is the clique number of Γ, namely the number of vertices of a maximum clique in Γ.

This polynomial has particularly elegant expressions for certain classes of graphs, such as complete bipartite and complete multipartite graphs. The following results illustrate this structure explicitly.

Proposition 2.11 (Fisher & Solow 1990)**.** Let Γ be a complete bipartite graph denoted by $\Gamma = K_{n_1,n_2}$. The clique polynomial of Γ is given by

$$C\left(K_{n_1,n_2}; x\right) = \left(1 + n_1 x\right)\left(1 + n_2 x\right).$$

Proposition 2.12 (Fisher & Solow 1990)**.** Let Γ be a complete k-partite graph denoted by $\Gamma = K_{n_1,n_2,\ldots,n_k}$. The clique polynomial of Γ is given by

$$C\left(K_{n_1,n_2,\ldots,n_k}; x\right) = \prod_{i=1}^{k}\left(1 + n_i x\right).$$

2.3. *Coprime Graph of the Generalised Quaternion Group*

We now define the main algebraic object of interest in this paper, the generalised quaternion group, which will serve as the foundation for constructing and analysing its associated coprime graph.

Definition 2.13 (Rotman 2015)**.** *Generalised quaternion group.* The generalised quaternion group Q_{4n} for an integer $n \geq 2$ has order $4n$ and is defined by the group presentation

$$Q_{4n} = \left\langle a, b \mid a^{2n} = e,\ a^n = b^2,\ b^{-1}ab = a^{-1}\right\rangle.$$

According to Nurhabibah *et al.* (2023), the elements of the group are also related by

$$a^i b = b a^{-i}, \quad \text{for } 1 \leq i \leq 2n,$$

and the order of $a^i b$ is always 4.

With the structure of the group established, we now recall a key result regarding the orders of elements in Q_{4n}, which will be crucial in determining adjacency in the coprime graph.

Lemma 2.14 (Zahidah *et al.* 2025)**.** *Let Q_{4n} be the generalised quaternion group. Then the order of its elements in the form of $a^i b^j$, where $a^i b^j \neq e$ and $0 \leq i < 2n$, is*

$$\left|a^i b^j\right| = \begin{cases} \dfrac{2n}{\gcd(i, 2n)}, & \textit{if } j = 0, \\ 4, & \textit{if } j = 1. \end{cases}$$

Equipped with this understanding of element orders, we proceed to define the coprime graph of a finite group, a graph whose adjacency condition depends on the greatest common divisor of the orders of its vertices.

Definition 2.15 (Ma *et al.* 2014)**.** *Coprime graph.* Let G be a finite group. The coprime graph of G is denoted by Γ_G^{cp}, in which every vertex of the graph represents an element of G. Any two distinct vertices u and v are adjacent in Γ_G^{cp} if and only if

$$\gcd\left(|u|, |v|\right) = 1,$$

where $|u|$ and $|v|$ are the orders of the elements u and v, respectively.

Using the above definitions and properties of Q_{4n}, we now state and prove a structural result about the multipartite nature and clique number of its coprime graph.

Corollary 2.16. *Let*

$$n = 2^{k_0} p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m},$$

where p_i are distinct odd primes, $k_0 \in \mathbb{N} \cup \{0\}$ and $k_i \in \mathbb{N}$ for $i = 1, 2, \ldots, m$. Then $\Gamma_{Q_{4n}}^{\text{cp}}$ is $(m+2)$-partite and

$$\omega\left(\Gamma_{Q_{4n}}^{\text{cp}}\right) = m + 2.$$

In particular, when $m = 0$, that is, when $n = 2^{k_0}$, the graph $\Gamma_{Q_{4n}}^{\text{cp}}$ is complete bipartite and

$$\omega\left(\Gamma_{Q_{4n}}^{\text{cp}}\right) = 2.$$

Proof. The structure of the coprime graph $\Gamma_{Q_{4n}}^{\text{cp}}$ has been analysed in previous works by Gayatri *et al.* (2023) and Nurhabibah *et al.* (2021), where four distinct cases are identified based on the factorisation of n. These are summarised as follows:

- When $n = 2^k$, the graph is complete bipartite and
$$\omega\left(\Gamma_{Q_{4n}}^{\text{cp}}\right) = 2.$$
- When n is an odd prime, the graph is tripartite and
$$\omega\left(\Gamma_{Q_{4n}}^{\text{cp}}\right) = 3.$$
- When $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ with $p_1 = 2$, the graph is $(m+1)$-partite and
$$\omega\left(\Gamma_{Q_{4n}}^{\text{cp}}\right) = m + 1.$$
- When $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ with $p_i \neq 2$, the graph is $(m+2)$-partite and
$$\omega\left(\Gamma_{Q_{4n}}^{\text{cp}}\right) = m + 2.$$

We now generalise these observations. Consider the case where n is composed entirely of odd prime factors, namely

$$n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}.$$

Let m denote the number of distinct odd prime factors of n. Then the function

$$f(m) = m + 2$$

gives the total number of partite sets in $\Gamma_{Q_{4n}}^{\text{cp}}$.

For the even case, suppose n includes the prime 2, say $p_1 = 2$. Then the number of odd prime factors is reduced to $m - 1$, and applying the same function yields

$$f(m-1) = (m-1) + 2 = m + 1,$$

which agrees with the known result for this case.

Therefore, all these cases can be unified under the general expression

$$n = 2^{k_0} p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m},$$

where p_i are distinct odd primes, $k_0 \in \mathbb{N} \cup \{0\}$ and $k_i \in \mathbb{N}$ for $i = 1, 2, \ldots, m$. In this setting, the coprime graph $\Gamma_{Q_{4n}}^{\mathrm{cp}}$ is $(m+2)$-partite, and its clique number is given by

$$\omega\left(\Gamma_{Q_{4n}}^{\mathrm{cp}}\right) = m + 2.$$

Finally, when $m = 0$, we have $n = 2^{k_0}$. Hence, the graph becomes complete bipartite with

$$\omega\left(\Gamma_{Q_{4n}}^{\mathrm{cp}}\right) = 2.$$

This recovers the first case and completes the generalisation. □

3. Clique Polynomials of the Coprime Graph of the Generalised Quaternion Group

This section explores the orders of elements in the generalised quaternion group Q_{4n} and explains how Euler's totient function is applied in this context. Several key notions and notations are introduced and used to compute the clique polynomials of the coprime graph $\Gamma_{Q_{4n}}^{\mathrm{cp}}$ for small cases of the prime factorisation of n. Specifically, we consider values of n with up to six distinct prime factors.

Corollary 3.1. *Let $a^i \in Q_{4n}$, where $1 \leq i \leq 2n$. Then $\left|a^i\right|$ divides $2n$.*

Proof. From Lemma 2.14, we have

$$\left|a^i\right| = \frac{2n}{\gcd(i, 2n)}.$$

Equivalently,

$$\gcd(i, 2n) = \frac{2n}{|a^i|}.$$

Since $\gcd(i, 2n)$ is an integer, it follows that $\left|a^i\right|$ divides $2n$. □

To facilitate the analysis of the coprime graph of the generalised quaternion group Q_{4n}, we introduce several notions and notations that will be used throughout this paper. These definitions are centred on classifying the elements of Q_{4n} based on their orders, which is a key ingredient in determining adjacency in the coprime graph. We define the following sets:

- The set of elements in Q_{4n} with order exactly x is denoted by O_x, defined as

  $$O_x = \{r \in Q_{4n} \mid |r| = x\}.$$

- For a positive integer k, the set of elements in Q_{4n} with order $x, x^2, \ldots, x^k$ is denoted by $O_{\{x;k\}}$, defined as

  $$O_{\{x;k\}} = \bigcup_{i=1}^{k} O_{x^i}.$$

- More generally, for positive integers $x_1, \ldots, x_m$ and $k_1, \ldots, k_m$, the set of elements in Q_{4n} with order of the form $x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}$, where $1 \leq i_j \leq k_j$ for each j, is denoted by $O_{\{x_1;k_1\}\{x_2;k_2\}\cdots\{x_m;k_m\}}$, defined as

$$O_{\{x_1;k_1\}\{x_2;k_2\}\cdots\{x_m;k_m\}} = \bigcup_{\substack{1\leq i_j \leq k_j \\ j=1,\ldots,m}} O_{x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}}.$$

These sets allow us to group elements of Q_{4n} according to their orders and provide a structured way to analyse their contributions to the structure of the coprime graph. The next result gives the size of these sets in terms of Euler's totient function $\phi(x)$, which plays a central role in our analysis.

Proposition 3.2. Let $O_x \subset Q_{4n}$, where x is a positive divisor of $2n$. Then

$$|O_x| = \begin{cases} \phi(x), & x \neq 4, \\ 2(n+1), & x = 4,\ n \text{ is even}, \\ 2n, & x = 4,\ n \text{ is odd}. \end{cases} \tag{2}$$

Proof. Consider the elements of Q_{4n} in the form a^i for $1 \leq i \leq 2n$. These elements form the cyclic subgroup $\langle a \rangle$ of order $2n$. Let $a^m \in \langle a \rangle$ be such that $a^{mx} = e$. By Lemma 2.14, the order of a^m is given by

$$|a^m| = \frac{2n}{\gcd(m, 2n)}.$$

Thus, a^m generates a cyclic subgroup $\langle a^m \rangle$ of order x. According to Theorem 2.2, the number of generators of this subgroup is $\phi(x)$, and these are precisely the elements a^m with $\gcd(m, x) = 1$.

Since $|a^m| = x$, it follows that $a^m \in O_x$. Therefore, for $x \neq 4$, we obtain

$$|O_x| = |\text{gen}\,(\langle a^m \rangle)| = \phi(x).$$

Now consider the case $x = 4$. In this case, we must also include elements of the form $a^i b \in Q_{4n}$, which, by Definition 2.13, have order exactly 4.

If n is even, say $n = 2l$ for some integer l, then $2n = 4l$, and hence 4 divides $2n$. Thus, there exist elements $a^i \in \langle a \rangle$ such that $|a^i| = 4$. Specifically, there are $\phi(4) = 2$ such elements.

In addition, there are $2n$ elements of the form $a^i b$, each of which has order 4. Hence, the total number of elements of order 4 is

$$|O_4| = \phi(4) + 2n = 2 + 2n = 2(n+1).$$

On the other hand, if n is odd, then 4 does not divide $2n$, so no element of the form a^i has order 4. However, the $2n$ elements of the form $a^i b$ still have order 4, as guaranteed by Definition 2.13. Thus, in this case,

$$|O_4| = 2n.$$

This completes the proof. □

Proposition 3.3. Let $O_{\{x;k\}} \subset Q_{4n}$. Then

$$\left|O_{\{x;k\}}\right| = \begin{cases} x^k - 1, & \text{when } x \text{ is an odd prime,} \\ 2n + 2^k - 1, & \text{when } x = 2. \end{cases} \tag{3}$$

Proof. Suppose first that x is an odd prime. Since the sets $O_x, O_{x^2}, \ldots, O_{x^k}$ are pairwise disjoint, the cardinality of $O_{\{x;k\}}$ is the sum of the cardinalities of its constituent sets. By Proposition 3.2, we have

$$\begin{aligned} \left|O_{\{x;k\}}\right| &= \left|\bigcup_{\ell=1}^{k} O_{x^\ell}\right| \\ &= |O_x| + |O_{x^2}| + \cdots + |O_{x^k}| \\ &= \phi(x) + \phi(x^2) + \cdots + \phi(x^k) \\ &= (x-1) + (x^2 - x) + (x^3 - x^2) + \cdots + (x^k - x^{k-1}) \\ &= x^k - 1. \end{aligned}$$

Next, consider the case $x = 2$. In the generalised quaternion group Q_{4n}, elements of order 4 include both elements from the cyclic subgroup $\langle a \rangle$ and the $2n$ elements of the form $a^i b$. Hence, the contribution from elements of order 4 requires special consideration.

For the powers of 2 arising from the cyclic subgroup $\langle a \rangle$, the sum of the Euler totient values is

$$\sum_{\ell=1}^{k} \phi(2^\ell) = 1 + 2 + 4 + \cdots + 2^{k-1} = 2^k - 1.$$

In addition, there are $2n$ elements of the form $a^i b$, each of order 4. Therefore,

$$\left|O_{\{2;k\}}\right| = 2n + \sum_{\ell=1}^{k} \phi(2^\ell) = 2n + 2^k - 1.$$

This completes the proof. □

Proposition 3.4. Let

$$O_{\{x_1;k_1\}\{x_2;k_2\}\cdots\{x_m;k_m\}} \subset Q_{4n},$$

where $x_1, x_2, \ldots, x_m$ are distinct odd prime numbers and $k_j \in \mathbb{N}$ for $j = 1, 2, \ldots, m$. Then

$$\left|O_{\{x_1;k_1\}\{x_2;k_2\}\cdots\{x_m;k_m\}}\right| = \prod_{i=1}^{m} \left|O_{\{x_i;k_i\}}\right|. \tag{4}$$

Proof. By definition,

$$O_{\{x_1;k_1\}\{x_2;k_2\}\cdots\{x_m;k_m\}} = \bigcup_{\substack{1 \le i_j \le k_j \\ j=1,\ldots,m}} O_{x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}}.$$

Since $x_1, x_2, \ldots, x_m$ are distinct odd primes, the integers

$$x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}$$

are distinct for different choices of $(i_1, i_2, \ldots, i_m)$. Hence, the sets in the above union are pairwise disjoint. Therefore,

$$\begin{aligned}\left|O_{\{x_1;k_1\}\{x_2;k_2\}\cdots\{x_m;k_m\}}\right| &= \sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_2}\cdots\sum_{i_m=1}^{k_m}\left|O_{x_1^{i_1}x_2^{i_2}\cdots x_m^{i_m}}\right| \\ &= \sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_2}\cdots\sum_{i_m=1}^{k_m}\phi\left(x_1^{i_1}x_2^{i_2}\cdots x_m^{i_m}\right).\end{aligned}$$

Using the multiplicative property of Euler's totient function for relatively prime integers, we obtain

$$\begin{aligned}\left|O_{\{x_1;k_1\}\{x_2;k_2\}\cdots\{x_m;k_m\}}\right| &= \sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_2}\cdots\sum_{i_m=1}^{k_m}\prod_{r=1}^{m}\phi\left(x_r^{i_r}\right) \\ &= \prod_{r=1}^{m}\left(\sum_{i_r=1}^{k_r}\phi\left(x_r^{i_r}\right)\right) \\ &= \prod_{r=1}^{m}\left|O_{\{x_r;k_r\}}\right|.\end{aligned}$$

This completes the proof. □

We begin by presenting the clique polynomial of the coprime graph of Q_{4n} for small factorisations of n, where the structure of the group leads to simpler expressions. Specifically, we have the following result.

Theorem 3.5. *The clique polynomial of the coprime graph of Q_{4n} is given as follows:*

(1) When $n = 2^k$, $k \in \mathbb{N}$,

$$C\left(\Gamma^{\mathrm{cp}}_{Q_{4n}}; x\right) = (1+x)(1+(4n-1)x). \tag{5}$$

(2) When $n = 2^{k_0}p_1^{k_1}$, where p_1 is an odd prime, $k_0 \in \mathbb{N}\cup\{0\}$ and $k_1 \in \mathbb{N}$, let

$$\alpha_1 = p_1^{k_1} - 1 \quad \textit{and} \quad \beta_0 = 2n + 2^{k_0+1} - 1.$$

Then

$$C\left(\Gamma^{\mathrm{cp}}_{Q_{4n}}; x\right) = 1 + c_1x + c_2x^2 + c_3x^3, \tag{6}$$

where

$$c_1 = 4n, \qquad c_2 = (4n-1) + \beta_0\alpha_1, \qquad c_3 = \beta_0\alpha_1.$$

Proof.

(1) By Corollary 2.16, when $n = 2^k$, the coprime graph $\Gamma^{\mathrm{cp}}_{Q_{4n}}$ is a complete bipartite graph. Since $2n = 2^{k+1}$, the possible orders arising from the cyclic subgroup $\langle a\rangle$ are powers of

2, namely

$$\left\{1, 2, 4, \ldots, 2^{k+1}\right\}.$$

The identity element e has order 1, while all remaining elements of Q_{4n} belong to $O_{\{2;k+1\}}$. Since the identity element is coprime to every other element, the graph is isomorphic to the star graph $K_{1,4n-1}$. Hence, by Proposition 2.11, the clique polynomial is

$$C\left(\Gamma^{\text{cp}}_{Q_{4n}}; x\right) = (1+x)\left(1+(4n-1)x\right).$$

(2) Let $n = 2^{k_0} p_1^{k_1}$, where p_1 is an odd prime. By listing the factors of

$$2n = 2^{k_0+1} p_1^{k_1},$$

the elements of Q_{4n} can be partitioned according to their orders into the following sets:

$$\left\{\{e\}, O_{\{2;k_0+1\}}, O_{\{p_1;k_1\}}, O_{\{2;k_0+1\}\{p_1;k_1\}}\right\}.$$

Let

$$A = O_{\{2;k_0+1\}}, \qquad B = O_{\{p_1;k_1\}}, \qquad D = O_{\{2;k_0+1\}\{p_1;k_1\}}.$$

Then

$$|\{e\}| + |A| + |B| + |D| = 4n.$$

Hence,

$$|A| + |B| + |D| = 4n - 1.$$

By Proposition 3.3, we have

$$|A| = 2n + 2^{k_0+1} - 1 \quad \text{and} \quad |B| = p_1^{k_1} - 1 = \alpha_1.$$

The coefficient c_1 counts all vertices of the graph. Therefore,

$$c_1 = 4n.$$

The coefficient c_2 counts all cliques of size 2, namely all edges in the graph. Since the identity element is adjacent to every non-identity element, it contributes $4n-1$ edges. In addition, every element in A is adjacent to every element in B, since their orders are coprime. Thus,

$$\begin{aligned} c_2 &= (4n-1) + |A||B| \\ &= (4n-1) + \left(2n + 2^{k_0+1} - 1\right)\alpha_1. \end{aligned}$$

The coefficient c_3 counts all cliques of size 3. Such cliques consist of the identity element, one element from A, and one element from B. Therefore,

$$c_3 = |\{e\}||A||B|$$

$$= \left(2n + 2^{k_0+1} - 1\right) \alpha_1.$$

This completes the proof. □

As the number of distinct primes in the prime factorisation of n increases, the listing of subsets becomes tedious and time-consuming. Therefore, another approach is proposed, where

$$\alpha_i = \left|O_{\{p_i;k_i\}}\right| = p_i^{k_i} - 1, \quad \text{for } k_i \in \mathbb{N} \text{ and } i = 1, 2, \ldots, m.$$

Inserting the value of each size-k clique into the clique polynomial shortens the expression and simplifies the calculation.

By using this method, determining the clique polynomial for the prime factorisation of n up to six distinct prime numbers with their respective indices can be shown in the following theorem. We begin by listing the independent vertex sets, which are also vertex-disjoint graphs, based on the factors of $2n$. This is followed by selecting the union of the joins of the graphs according to the size of the clique. Finally, by applying (2), (3) and (4), each coefficient is obtained for the clique polynomial up to index $m + 2$.

Theorem 3.6. *The clique polynomial of the coprime graph of Q_{4n} is given as follows:*

(1) When $n = 2^{k_0} p_1^{k_1} p_2^{k_2}$, where p_1 and p_2 are distinct odd primes, $k_0 \in \mathbb{N} \cup \{0\}$, $k_i \in \mathbb{N}$, and $\alpha_i = p_i^{k_i} - 1, \quad i = 1, 2$, let

$$\beta_0 = 2n + 2^{k_0+1} - 1 \quad \textit{and} \quad \lambda_0 = 2^{k_0+1}.$$

Then

$$C\left(\Gamma^{\mathrm{cp}}_{Q_{4n}}; x\right) = 1 + c_1x + c_2x^2 + c_3x^3 + c_4x^4, \tag{7}$$

where

$$\begin{aligned} c_1 &= 4n, \\ c_2 &= 4n - 1 + \beta_0(\alpha_1 + \alpha_2) + (2\lambda_0 - 1 + \beta_0)\alpha_1\alpha_2, \\ c_3 &= \beta_0(\alpha_1 + \alpha_2) + (2\lambda_0 - 1 + 2\beta_0)\alpha_1\alpha_2, \\ c_4 &= \beta_0\alpha_1\alpha_2. \end{aligned}$$

(2) When

$$n = 2^{k_0} p_1^{k_1} p_2^{k_2} p_3^{k_3},$$

where p_1, p_2 and p_3 are distinct odd primes, $k_0 \in \mathbb{N} \cup \{0\}$, $k_i \in \mathbb{N}$, and

$$\alpha_i = p_i^{k_i} - 1, \quad i = 1, 2, 3,$$

let

$$\beta_0 = 2n + 2^{k_0+1} - 1 \quad \textit{and} \quad \lambda_0 = 2^{k_0+1}.$$

Then

$$C\left(\Gamma^{\mathrm{cp}}_{Q_{4n}}; x\right) = 1 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5, \tag{8}$$

where

$$\begin{aligned}
c_1 &= 4n,\\
c_2 &= 4n - 1 + \beta_0(\alpha_1 + \alpha_2 + \alpha_3)\\
&\quad + (2\lambda_0 - 1 + \beta_0)(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)\\
&\quad + (6\lambda_0 - 3 + \beta_0)\alpha_1\alpha_2\alpha_3,\\
c_3 &= \beta_0(\alpha_1 + \alpha_2 + \alpha_3)\\
&\quad + (2\lambda_0 - 1 + 2\beta_0)(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)\\
&\quad + (9\lambda_0 - 5 + 4\beta_0)\alpha_1\alpha_2\alpha_3,\\
c_4 &= \beta_0(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)\\
&\quad + (3\lambda_0 - 2 + 4\beta_0)\alpha_1\alpha_2\alpha_3,\\
c_5 &= \beta_0\alpha_1\alpha_2\alpha_3.
\end{aligned}$$

(3) When

$$n = 2^{k_0}p_1^{k_1}p_2^{k_2}p_3^{k_3}p_4^{k_4},$$

where p_1, p_2, p_3 and p_4 are distinct odd primes, $k_0 \in \mathbb{N} \cup \{0\}$, $k_i \in \mathbb{N}$, and

$$\alpha_i = p_i^{k_i} - 1, \quad i = 1, 2, 3, 4,$$

let

$$\beta_0 = 2n + 2^{k_0+1} - 1 \quad \textit{and} \quad \lambda_0 = 2^{k_0+1}.$$

Then

$$C\left(\Gamma^{\mathrm{cp}}_{Q_{4n}}; x\right) = 1 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6, \tag{9}$$

where

$$\begin{aligned}
c_1 &= 4n,\\
c_2 &= 4n - 1 + \beta_0(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\\
&\quad + (2\lambda_0 - 1 + \beta_0)\sum_{1\le i<j\le 4}\alpha_i\alpha_j\\
&\quad + (6\lambda_0 - 3 + \beta_0)\sum_{1\le i<j<k\le 4}\alpha_i\alpha_j\alpha_k\\
&\quad + (14\lambda_0 - 7 + \beta_0)\alpha_1\alpha_2\alpha_3\alpha_4,\\
c_3 &= \beta_0(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\\
&\quad + (2\lambda_0 - 1 + 2\beta_0)\sum_{1\le i<j\le 4}\alpha_i\alpha_j\\
&\quad + (9\lambda_0 - 5 + 4\beta_0)\sum_{1\le i<j<k\le 4}\alpha_i\alpha_j\alpha_k\\
&\quad + (32\lambda_0 - 19 + 8\beta_0)\alpha_1\alpha_2\alpha_3\alpha_4,\\
c_4 &= \beta_0\sum_{1\le i<j\le 4}\alpha_i\alpha_j
\end{aligned}$$

$$
\begin{aligned}
&+ (3\lambda_0 - 2 + 4\beta_0) \sum_{1 \le i < j < k \le 4} \alpha_i \alpha_j \alpha_k \\
&+ (22\lambda_0 - 15 + 13\beta_0)\alpha_1\alpha_2\alpha_3\alpha_4, \\
c_5 &= \beta_0 \sum_{1 \le i < j < k \le 4} \alpha_i \alpha_j \alpha_k \\
&+ (4\lambda_0 - 3 + 7\beta_0)\alpha_1\alpha_2\alpha_3\alpha_4, \\
c_6 &= \beta_0\alpha_1\alpha_2\alpha_3\alpha_4.
\end{aligned}
$$

4. Odd-Even Clique Equality

This section presents the main result of the paper: an equality between the total number of odd-sized and even-sized cliques in the coprime graph of the generalised quaternion group Q_{4n}. This parity balance arises from the structure of the clique polynomial and reveals a deep symmetry in the combinatorial properties of the graph.

By Corollary 2.16, the case when $n = 2^{k_0}$ is considered trivial. From the proof of (5), the corresponding coprime graph is either a complete bipartite graph or a star graph. In contrast, (6)–(9) demonstrate how the clique polynomials are constructed for increasing complexity in the prime factorisation of n, revealing a consistent pattern in the coefficients.

In any graph, the size-0 clique corresponds to the empty set, and the size-1 cliques correspond to the individual vertices. Therefore, for the coprime graph $\Gamma^{\text{cp}}_{Q_{4n}}$, we have

$$c_0 = 1 \quad \text{and} \quad c_1 = 4n.$$

Hence, our focus shifts to cliques of size two and above.

An interesting pattern emerges when comparing the alternating sums of clique counts across different values of n, starting from the trivial case $n = 2^{k_0}$. For increasing numbers of distinct prime factors in n, we observe the following relationships:

$$
\begin{cases}
n = 2^{k_0}, & c_0 - c_1 + c_2 = 0, \\
n = 2^{k_0} p_1^{k_1}, & c_0 - c_1 + c_2 - c_3 = 0, \\
n = 2^{k_0} p_1^{k_1} p_2^{k_2}, & c_0 - c_1 + c_2 - c_3 + c_4 = 0, \\
n = 2^{k_0} p_1^{k_1} p_2^{k_2} p_3^{k_3}, & c_0 - c_1 + c_2 - c_3 + c_4 - c_5 = 0, \\
n = 2^{k_0} p_1^{k_1} p_2^{k_2} p_3^{k_3} p_4^{k_4}, & c_0 - c_1 + c_2 - c_3 + c_4 - c_5 + c_6 = 0.
\end{cases}
$$

This pattern suggests that the numbers of odd-sized and even-sized cliques cancel each other under alternating summation. This key insight supports our main result, Theorem 1.1. We now proceed with the formal proof.

Proof. of Theorem 1.1 Since the identity element e has order 1, it is adjacent to every other vertex in $\Gamma^{\text{cp}}_{Q_{4n}}$. Hence, any clique that does not contain e can be extended to a larger clique by adding e.

Let c_k denote the number of size-k cliques, c_k^e the number of size-k cliques that contain e, and c_k' the number of size-k cliques that do not contain e. Then

$$c_k = c_k^e + c_k'.$$

Moreover, adding the identity element e to any size-k clique that does not contain e gives a size-$(k+1)$ clique that contains e. Conversely, removing e from any size-$(k+1)$ clique that contains e gives a size-k clique that does not contain e. Therefore,

$$c'_k = c^e_{k+1}.$$

Also, the empty clique does not contain e, and no maximum clique can be extended further by adding e. Thus,

$$c^e_0 = 0 \quad \text{and} \quad c'_\omega = 0.$$

From Corollary 2.16, the clique number of $\Gamma^{\mathrm{cp}}_{Q_{4n}}$ is $\omega = m+2$, where $m \in \mathbb{N}\cup\{0\}$. Consider the alternating sum over all clique sizes up to ω:

$$\begin{aligned}
\sum_{k=0}^{\omega}(-1)^k c_k &= \sum_{k=0}^{\omega}(-1)^k \left(c^e_k + c'_k\right) \\
&= \sum_{k=0}^{\omega}(-1)^k \left(c^e_k + c^e_{k+1}\right) \\
&= c^e_0 + \sum_{k=1}^{\omega}\left((-1)^k + (-1)^{k-1}\right) c^e_k + (-1)^\omega c'_\omega \\
&= c^e_0 + (-1)^\omega c'_\omega \\
&= 0.
\end{aligned}$$

Hence,

$$\sum_{k=0}^{\omega}(-1)^k c_k = 0.$$

Equivalently,

$$\sum_{j=0}^{\lfloor \frac{\omega}{2} \rfloor} c_{2j} = \sum_{j=0}^{\lfloor \frac{\omega-1}{2} \rfloor} c_{2j+1}.$$

Thus, the total number of even-sized cliques is equal to the total number of odd-sized cliques in $\Gamma^{\mathrm{cp}}_{Q_{4n}}$. This completes the proof. □

5. Conclusion

This paper presents the general forms of the clique polynomials of the coprime graph of Q_{4n} for selected cases of prime factorisation, as given in (5), (6), (7), (8) and (9). A systematic method is proposed to simplify and shorten the calculation of clique polynomials for larger cases of prime factorisation. Furthermore, the odd-even clique equality for the coprime graph associated with Q_{4n} is established in Theorem 1.1. This underexplored property may be related to the symmetric structure of the coprime graph and can be further investigated in future studies. Further applications of this clique property to other graphs are also suggested, which may contribute to the development of new classes of graphs in graph theory.

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References

Alaeiyan M. & Mohammadian S. 2011. Graph polynomials. *International Scholarly Research Notices Algebra* **2011**(1): 116–125.

Balakrishnan R. & Ranganathan K. 2012. *A Textbook of Graph Theory*. New York: Springer.

Bazin A. & Bertaux A. 2018. k-Partite graphs as contexts. *Proceedings of the 14th International Conference on Concept Lattices and Their Applications*. CEUR Workshop Proceedings.

Fisher D. & Solow A. 1990. Dependence polynomials. *Discrete Mathematics* **82**(3): 251 – 258.

Gallian J.A. 2020. *Contemporary Abstract Algebra*. Boca Raton: CRC Press.

Gayatri M.R., Nurhabibah N., Aini Q., Awanis Z.Y., Salwa S. & Wardhana I.G.A.W. 2023. The clique number and the chromatics number of the coprime graph for the generalized quaternion group. *JTAM (Jurnal Teori dan Aplikasi Matematika)* **7**(2): 409 – 416.

Gross J.L. & Tucker T.W. 2001. *Topological Graph Theory*. Mineola, New York: Courier Corporation.

Hoede C. & Li X. 1994. Clique polynomials and independent set polynomials of graphs. *Discrete Mathematics* **125**(1-3): 219 – 228.

Knill O. 2021. Graph complements of circular graphs. arXiv preprint arXiv:2101.06873.

Lehmer D.H. 1932. On Euler's totient function. *Bulletin of the American Mathematical Society* **38**(10): 745 – 751.

Ma X., Wei H. & Yang L. 2014. The coprime graph of a group. *International Journal of Group Theory* **3**(3): 13–23.

Munandar A. 2023. Some properties on coprime graph of generalized quaternion groups. *BAREKENG: Jurnal Ilmu Matematika dan Terapan* **17**(3): 1373 – 1380.

Najmuddin N. 2021. Graph polynomial associated to some graphs of certain finite nonabelian groups. Tesis PhD, Universiti Teknologi Malaysia.

Nurhabibah N., Syarifudin A.G. & Wardhana I.G.A.W. 2021. Some results of the coprime graph of a generalized quaternion group Q_{4n}. *Prime: Indonesian Journal of Pure and Applied Mathematics* **3**(1): 29 – 33.

Nurhabibah N., Wardhana I.G.A.W. & Switrayni N.W. 2023. Numerical invariants of coprime graph of a generalized quaternion group. *Journal of the Indonesian Mathematical Society* **29**(1): 36–44.

Rosen K.H. 2019. *Elementary Number Theory*. Boston: Pearson.

Rotman J.J. 2015. *Advanced Modern Algebra*. Providence, Rhode Island: American Mathematical Society.

Zahidah S., Mahanani D.M. & Oktaviana K.L. 2021. Connectivity indices of coprime graph of generalized quaternion group. *Journal of the Indonesian Mathematical Society* **27**(3): 285 – 296.

Zahidah S., Zulfikar A.N., Estuningsih N. & Erfanian A. 2025. Representation matrices of coprime graph of generalized quaternion group. *Journal of the Indonesian Mathematical Society* **31**(3): 1772–1794.

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