

GEGENBAUER POLYNOMIALS AND INEQUALITY FOR ANALYTIC BI-UNIVALENT FUNCTIONS

(Polinomial Gegenbauer dan Ketaksamaan bagi Fungsi Analitik Bi-univalen)

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ABSTRACT

This paper introduces a new subclass of analytic bi-univalent functions related to the Gegenbauer polynomial. The coefficient bounds for the second and third terms of the functions belonging to this subclass are studied. The Fekete-Szegő problem is also obtained. Furthermore, several new results are obtained by specialising the parameters involved in our main results. The methods used are the convolution concept and rigorous proof techniques.

Keywords: Gegenbauer polynomials; bi-univalent functions; analytic functions; Fekete-Szegő problem

ABSTRAK

Artikel ini memperkenalkan subkelas baharu bagi fungsi analitik bi-univalen yang berkaitan dengan polinomial Gegenbauer. Batas pekali bagi sebutan kedua dan ketiga bagi fungsi yang tergolong dalam subkelas tersebut dikaji. Masalah Fekete-Szegő juga diperolehi. Tambahan pula, beberapa hasil baharu diperolehi apabila parameter tertentu yang terlibat dalam hasil utama dikhususkan. Kaedah yang digunakan ialah konsep konvolusi dan teknik pembuktian yang jitu.

Kata kunci: polinomial Gegenbauer; fungsi bi-univalen; fungsi analitik; masalah Fekete-Szegő

1. Introduction

Let $\mathcal{A}$ be the class of regular or analytic functions f of the form

$$f(z) = z + \rho_2 z^2 + \rho_3 z^3 + \dots, \quad (1)$$

which are defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalised conditions $f(0) = 0$ and $f'(0) = 1$. Further, denote by S the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. The well-known Koebe one-quarter theorem by Duren (1983) states that if $f \in S$, then f has an inverse f^{-1} defined by $f^{-1}(f(z)) = z$, ($z \in \mathbb{U}$), and $f(f^{-1}(w)) = w$, ($|w| < r_0(f)$, $r_0(f) \geq \frac{1}{4}$), where

$$f^{-1}(w) = w - \rho_2 w^2 + (2\rho_2^2 - \rho_3) w^3 - (5\rho_2^3 - 5\rho_2\rho_3 + \rho_4) w^4 + \dots = g(w). \quad (2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in the open unit disk $\mathbb{U}$ if both f and its inverse f^{-1} are univalent in $\mathbb{U}$. The class of all such bi-univalent functions is denoted by Σ .

More precisely, let Σ denote the class of analytic bi-univalent functions in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, which can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} \rho_n z^n = z + \rho_2 z^2 + \rho_3 z^3 + \dots,$$

where both f and its inverse f^{-1} are univalent in $\mathbb{U}$.

In recent years, studies on analytic bi-univalent functions have been revisited by several researchers (Ahmad *et al.* 2022; Rehman *et al.* 2021; Frasin *et al.* 2020; Srivastava *et al.* 2010; Yousef *et al.* 2019; Alnajjar *et al.* 2024a,b) due to their fundamental role in geometric function theory and their applications in various mathematical and engineering problems. This renewed interest is also fuelled by new analytical tools, such as special functions, fractional calculus and linear operators, which allow a deeper study of coefficient problems and the characteristics of these functions. In addition, recent studies involving nonlinear fractional models and large-scale optimisation methods further indicate the growing importance of advanced analytical and computational techniques in mathematical research (Eshkuvatov *et al.* 2025; Alhawarat *et al.* 2021).

Legendre first introduced orthogonal polynomials in 1784 (Legendre 1785). Most of the time, orthogonal polynomials are used to solve ordinary differential equations with certain boundary conditions. Orthogonal polynomials are also very important in approximation theory; for more details, see (Bateman 1953; Doman 2015).

The Gegenbauer polynomials $\perp_\alpha(x, \aleph)$, studied in (Amourah *et al.* 2021b,c), are stated as

$$\perp_\alpha(x, \aleph) = \frac{1}{(1 - 2x\aleph + \aleph^2)^\alpha}, \quad (3)$$

where, in this case, $x \in [-1, 1]$ and $\aleph \in \mathbb{U}$. Since $\perp_\alpha$ is regular in $\mathbb{U}$ for fixed x , it can be expanded in a Taylor series as

$$\perp_\alpha(x, \aleph) = \sum_{n=0}^{\infty} \top_n^\alpha(x) \aleph^n. \quad (4)$$

The Gegenbauer polynomial of degree n is denoted by the expression $\top_n^\alpha(x)$. On the other hand, the generating function of the Gegenbauer polynomial can also be expressed as

$$\perp_0(x, \aleph) = \sum_{n=0}^{\infty} \top_n^0(x) \aleph^n = 1 - \log(1 - 2x\aleph + \aleph^2), \quad (5)$$

when α is equal to zero. Furthermore, it is important to note that it is desired to normalise the value of α to a value greater than $-\frac{1}{2}$ (Doman 2015; Reimer 2012). It is not difficult to observe that the following recursive relation is satisfied by the Gegenbauer polynomials:

$$\top_n^\alpha(x) = \frac{1}{n} [2x(n + \alpha - 1) \top_{n-1}^\alpha(x) - (n + 2\alpha - 2) \top_{n-2}^\alpha(x)]. \quad (6)$$

The first three Gegenbauer polynomials are presented in the following order:

$$\top_0^\alpha(x) = 1, \quad \top_1^\alpha(x) = 2\alpha x, \quad \top_2^\alpha(x) = 2\alpha(1 + \alpha)x^2 - \alpha. \quad (7)$$

Special cases of Gegenbauer polynomials $\top_n^\alpha(x)$ are as follows:

- i) When the value of α is equal to one, the Chebyshev polynomials $\top_n^1(x)$ are obtained.
- ii) When the value of α is equal to 0.5, the Legendre polynomials $\top_n^{0.5}(x)$ are obtained.

These days, a multitude of researchers have focused their attention on the study of bi-univalent functions linked to orthogonal polynomials. A few of these researchers include (Amourah *et al.* 2021a, 2022; Kiepiela *et al.* 2003; Wanas & Yalçın 2021; Al-Hawary *et al.* 2021; Alnajjar & Darus 2024; Alnajjar *et al.* 2024c). Regarding Gegenbauer polynomials, a

limited number of works have explored their connection with bi-univalent functions. Notable contributions include those by (Alnajjar & Darus 2024).

We examine several new subclasses of bi-univalent functions that are subordinate to the Gegenbauer polynomial. Our goal is to obtain bounds for the $|\rho_2|$ and $|\rho_3|$ coefficients, as well as the Fekete-Szegő functional problems for functions that belong to these new classes.

2. Definition and Examples

This part starts with a definition of the family $\mathfrak{G}_\Sigma(\alpha, \zeta, x)$:

Definition 2.1. Let $0 \leq \zeta \leq 1$. A function $f \in \Sigma$ is said to be in the family $\mathfrak{G}_\Sigma(\alpha, \zeta, x)$ if it satisfies the subordinations

$$\frac{\aleph f'(\aleph) + \zeta \aleph^2 f''(\aleph)}{(1 - \zeta)f(\aleph) + \zeta \aleph f'(\aleph)} \prec \perp_\alpha(x, \aleph)$$

and

$$\frac{\beth g'(\beth) + \zeta \beth^2 g''(\beth)}{(1 - \zeta)g(\beth) + \zeta \beth g'(\beth)} \prec \perp_\alpha(x, \beth),$$

so that $x \in (\frac{1}{2}, 1]$, where $\perp_\alpha$ is the generating function of the Gegenbauer polynomial provided by equation (3), and the function $g(\beth) = f^{-1}(\beth)$ is defined by (2).

Example 2.2. Given that ζ is equal to zero, a function f belonging to the set Σ is said to be a member of the family $\mathfrak{G}_\Sigma(\alpha, 0, x)$ if it satisfies the subordinations

$$\frac{\aleph f'(\aleph)}{f(\aleph)} \prec \perp_\alpha(x, \aleph)$$

and

$$\frac{\beth g'(\beth)}{g(\beth)} \prec \perp_\alpha(x, \beth),$$

so that $x \in (\frac{1}{2}, 1]$, where $\perp_\alpha$ is the generating function of the Gegenbauer polynomial provided by equation (3), and the function $g(\beth) = f^{-1}(\beth)$ is defined by (2).

Example 2.3. Given that ζ is equal to one, a function f belonging to the set Σ is said to be a member of the family $\mathfrak{G}_\Sigma(\alpha, 1, x)$ if it satisfies the subordinations

$$1 + \frac{\aleph f''(\aleph)}{f'(\aleph)} \prec \perp_\alpha(x, \aleph)$$

and

$$1 + \frac{\beth g''(\beth)}{g'(\beth)} \prec \perp_\alpha(x, \beth),$$

so that $x \in (\frac{1}{2}, 1]$, where $\perp_\alpha$ is the generating function of the Gegenbauer polynomial provided by equation (3), and the function $g(\beth) = f^{-1}(\beth)$ is defined by (2).

3. Coefficient Bounds of the Class $\mathfrak{G}_\Sigma(\alpha, \zeta, x)$

The first thing that we will do is to provide the estimates of the coefficients for the class $\mathfrak{G}_\Sigma(\alpha, \zeta, x)$ that is described in Definition 2.1.

Lemma 3.1. *Let $\phi(\aleph) = c_1\aleph + c_2\aleph^2 + \dots$ and $\varrho(\beth) = d_1\beth + d_2\beth^2 + \dots$ be analytic functions in $\mathbb{U}$, such that $|\phi(\aleph)| < 1$ and $|\varrho(\beth)| < 1$ for all $\aleph, \beth \in \mathbb{U}$. Then*

$$|c_i| \leq 1 \quad \text{and} \quad |d_i| \leq 1 \quad \text{for all } i \in \mathbb{N}. \quad (8)$$

Theorem 3.2. *Let $f \in \mathfrak{G}_\Sigma(\alpha, \zeta, x)$ be given by (1), where $0 \leq \zeta \leq 1$. Then*

$$|\rho_2| \leq \frac{|2\alpha x| \sqrt{|2\alpha x|}}{\sqrt{|(2\alpha x)^2(1+2\zeta-\zeta^2) - (2\alpha(1+\alpha)x^2 - \alpha)(1+\zeta)^2|}},$$

and

$$|\rho_3| \leq \frac{\alpha x}{(1+2\zeta)} + \frac{(2\alpha x)^2}{(1+\zeta)^2}.$$

Proof. Let $f \in \mathfrak{G}_\Sigma(\alpha, \zeta, x)$. Then there are two analytic functions ϕ, ϱ given by

$$\phi(\aleph) = c_1\aleph + c_2\aleph^2 + c_3\aleph^3 + \dots \quad (\aleph \in \mathbb{U}) \quad (9)$$

and

$$\varrho(\beth) = d_1\beth + d_2\beth^2 + d_3\beth^3 + \dots \quad (\beth \in \mathbb{U}), \quad (10)$$

with $\phi(0) = \varrho(0) = 0$, $|\phi(\aleph)| < 1$, $|\varrho(\beth)| < 1$, $\aleph, \beth \in \mathbb{U}$, such that

$$\frac{\aleph f'(\aleph) + \zeta \aleph^2 f''(\aleph)}{(1-\zeta)f(\aleph) + \zeta \aleph f'(\aleph)} = \perp_\alpha(x, \phi(\aleph)) \quad (11)$$

and

$$\frac{\beth g'(\beth) + \zeta \beth^2 g''(\beth)}{(1-\zeta)g(\beth) + \zeta \beth g'(\beth)} = \perp_\alpha(x, \varrho(\beth)). \quad (12)$$

Combining (9), (10) and (11) yields

$$\frac{\aleph f'(\aleph) + \zeta \aleph^2 f''(\aleph)}{(1-\zeta)f(\aleph) + \zeta \aleph f'(\aleph)} = 1 + \top_1^\alpha(x)c_1\aleph + [\top_1^\alpha(x)c_2 + \top_2^\alpha(x)c_1^2]\aleph^2 + \dots \quad (13)$$

and

$$\frac{\beth g'(\beth) + \zeta \beth^2 g''(\beth)}{(1-\zeta)g(\beth) + \zeta \beth g'(\beth)} = 1 + \top_1^\alpha(x)d_1\beth + [\top_1^\alpha(x)d_2 + \top_2^\alpha(x)d_1^2]\beth^2 + \dots \quad (14)$$

When we compare the coefficients that correspond to equations (13) and (14), we end up with the following after simplifying the equations:

$$(1+\zeta)\rho_2 = \top_1^\alpha(x)c_1, \quad (15)$$

$$2(1+2\zeta)\rho_3 - (1+\zeta)^2\rho_2^2 = \top_1^\alpha(x)c_2 + \top_2^\alpha(x)c_1^2, \quad (16)$$

$$-(1 + \zeta)\rho_2 = \mathbb{T}_1^\alpha(x)d_1, \quad (17)$$

and

$$-2(1 + 2\zeta)\rho_3 - (\zeta^2 - 6\zeta - 3)\rho_2^2 = \mathbb{T}_1^\alpha(x)d_2 + \mathbb{T}_2^\alpha(x)d_1^2. \quad (18)$$

It follows from (15) and (17) that

$$c_1 = -d_1 \quad (19)$$

and

$$2(1 + \zeta)^2\rho_2^2 = [\mathbb{T}_1^\alpha(x)]^2 (c_1^2 + d_1^2). \quad (20)$$

If we add (16) to (18), we find that

$$2(1 + 2\zeta - \zeta^2)\rho_2^2 = \mathbb{T}_1^\alpha(x)(c_2 + d_2) + \mathbb{T}_2^\alpha(x)(c_1^2 + d_1^2). \quad (21)$$

By substituting the value of $c_1^2 + d_1^2$ from equation (20) into the right-hand side of equation (21), we are able to extract the following conclusion:

$$\rho_2^2 = \frac{[\mathbb{T}_1^\alpha(x)]^3 (c_2 + d_2)}{2[\mathbb{T}_1^\alpha(x)]^2 (1 + 2\zeta - \zeta^2) - 2\mathbb{T}_2^\alpha(x)(1 + \zeta)^2}. \quad (22)$$

Further computations using (7), (8) and (22) give

$$|\rho_2| \leq \frac{|2\alpha x| \sqrt{|2\alpha x|}}{\sqrt{|(2\alpha x)^2(1 + 2\zeta - \zeta^2) - (2\alpha(1 + \alpha)x^2 - \alpha)(1 + \zeta)^2|}}.$$

Next, if we subtract (18) from (16), we can easily see that

$$4(1 + 2\zeta)(\rho_3 - \rho_2^2) = \mathbb{T}_1^\alpha(x)(c_2 - d_2) + \mathbb{T}_2^\alpha(x)(c_1^2 - d_1^2). \quad (23)$$

In view of (19) and (20), we get from (23)

$$\rho_3 = \frac{\mathbb{T}_1^\alpha(x)(c_2 - d_2)}{4(1 + 2\zeta)} + \frac{[\mathbb{T}_1^\alpha(x)]^2 (c_1^2 + d_1^2)}{2(1 + \zeta)^2}.$$

Thus, applying (7), we obtain

$$|\rho_3| \leq \frac{\alpha x}{(1 + 2\zeta)} + \frac{(2\alpha x)^2}{(1 + \zeta)^2}.$$

This completes the proof of Theorem 3.2. $\square$

4. Fekete–Szegő Inequality for the Class $\mathfrak{G}_\Sigma(\alpha, \zeta, x)$

One of the well-known issues with the coefficients of univalent analytic functions is the Fekete–Szegő inequality (Fekete & Szegő 1933). Studying the Fekete–Szegő type inequality for the function in the class $\mathfrak{G}_\Sigma(\alpha, \zeta, x)$ is our goal in this section.

Theorem 4.1. Let $f \in \mathfrak{G}_\Sigma(\alpha, \zeta, x)$ be given by (1), where $0 \leq \zeta \leq 1$ and $\eta \in \mathbb{R}$. Then

$$|\rho_3 - \eta\rho_2^2| \leq \begin{cases} \frac{|\alpha x|}{1+2\zeta}, & |\eta - 1| \leq \ell, \\ \frac{|2\alpha x|^3 |1 - \eta|}{|(2\alpha x)^2(1+2\zeta - \zeta^2) - (2\alpha(1+\alpha)x^2 - \alpha)(1+\zeta)^2|}, & |\eta - 1| \geq \ell, \end{cases}$$

where

$$\ell = \left| \frac{(2\alpha x)^2(1+2\zeta - \zeta^2) - (2\alpha(1+\alpha)x^2 - \alpha)(1+\zeta)^2}{2(1+2\zeta)(2\alpha x)^2} \right|.$$

Proof. It follows from (22) and (23) that

$$\begin{aligned} \rho_3 - \eta\rho_2^2 &= \frac{\mathbb{T}_1^\alpha(x)(c_2 - d_2)}{4(1+2\zeta)} + (1-\eta)\rho_2^2 \\ &= \frac{\mathbb{T}_1^\alpha(x)(c_2 - d_2)}{4(1+2\zeta)} + \frac{[\mathbb{T}_1^\alpha(x)]^3(c_2 + d_2)(1-\eta)}{2[\mathbb{T}_1^\alpha(x)]^2(1+2\zeta - \zeta^2) - 2\mathbb{T}_2^\alpha(x)(1+\zeta)^2} \\ &= \mathbb{T}_1^\alpha(x) \left[\left(h(\eta) + \frac{1}{4(1+2\zeta)} \right) c_2 + \left(h(\eta) - \frac{1}{4(1+2\zeta)} \right) d_2 \right], \end{aligned}$$

where

$$h(\eta) = \frac{[\mathbb{T}_1^\alpha(x)]^2(1-\eta)}{2([\mathbb{T}_1^\alpha(x)]^2(1+2\zeta - \zeta^2) - \mathbb{T}_2^\alpha(x)(1+\zeta)^2)}.$$

According to (7), we find that

$$|\rho_3 - \eta\rho_2^2| \leq \begin{cases} \frac{|\mathbb{T}_1^\alpha(x)|}{2(1+2\zeta)}, & |h(\eta)| \leq \frac{1}{4(1+2\zeta)}, \\ 2|\mathbb{T}_1^\alpha(x)||h(\eta)|, & |h(\eta)| \geq \frac{1}{4(1+2\zeta)}. \end{cases}$$

This completes the proof of Theorem 4.1. $\square$

5. Corollaries

The novel corollaries and implications that follow are all derived from our major conclusions in this section.

Corollary 5.1. Let $f \in \mathfrak{G}_\Sigma(\alpha, 0, x)$ be given by (1). Then

$$\begin{aligned} |\rho_2| &\leq \frac{|2\alpha x| \sqrt{|2\alpha x|}}{\sqrt{|(2\alpha x)^2 - (2\alpha(1+\alpha)x^2 - \alpha)|}}, \\ |\rho_3| &\leq \alpha x + (2\alpha x)^2, \end{aligned}$$

and

$$|\rho_3 - \eta\rho_2^2| \leq \begin{cases} |\alpha x|, & |\eta - 1| \leq \left| \frac{(2\alpha x)^2 - (2\alpha(1 + \alpha)x^2 - \alpha)}{2(2\alpha x)^2} \right|, \\ \frac{|2\alpha x|^3 |1 - \eta|}{|(2\alpha x)^2 - (2\alpha(1 + \alpha)x^2 - \alpha)|}, & |\eta - 1| \geq \left| \frac{(2\alpha x)^2 - (2\alpha(1 + \alpha)x^2 - \alpha)}{2(2\alpha x)^2} \right|. \end{cases}$$

Corollary 5.2. Let $f \in \mathfrak{G}_\Sigma(\alpha, 1, x)$ be given by (1). Then

$$|\rho_2| \leq \frac{|2\alpha x| \sqrt{|2\alpha x|}}{\sqrt{|2(2\alpha x)^2 - 4(2\alpha(1 + \alpha)x^2 - \alpha)|}},$$

$$|\rho_3| \leq \frac{\alpha x}{3} + (\alpha x)^2,$$

and

$$|\rho_3 - \eta\rho_2^2| \leq \begin{cases} \frac{|\alpha x|}{3}, & |\eta - 1| \leq \left| \frac{(2\alpha x)^2 - 2(2\alpha(1 + \alpha)x^2 - \alpha)}{3(2\alpha x)^2} \right|, \\ \frac{|2\alpha x|^3 |1 - \eta|}{2|(2\alpha x)^2 - 2(2\alpha(1 + \alpha)x^2 - \alpha)|}, & |\eta - 1| \geq \left| \frac{(2\alpha x)^2 - 2(2\alpha(1 + \alpha)x^2 - \alpha)}{3(2\alpha x)^2} \right|. \end{cases}$$

Remark 5.3. Several new findings would be obtained for the classes $\mathfrak{G}_\Sigma(\frac{1}{2}, \zeta, x)$ for Legendre polynomials and $\mathfrak{G}_\Sigma(1, \zeta, x)$ for Chebyshev polynomials if the results described in this article were to be implemented.

6. Conclusion

This paper's main goal is to derive estimates for the Fekete-Szegő inequality $|\rho_3 - \eta\rho_2^2|$ and the Taylor-Maclaurin coefficients, specifically $|\rho_2|$ and $|\rho_3|$, for functions in the newly defined class $\mathfrak{G}_\Sigma(\alpha, \zeta, x)$ of analytic bi-univalent functions in the open unit disk $\mathbb{U}$.

The structure of Gegenbauer polynomials and subordination techniques were used in this study to successfully establish upper bounds for these coefficients and functionals. The acquired results demonstrate the versatility of creating subclasses of bi-univalent functions.

An intriguing open problem is to determine the sharp upper bounds for the stated coefficients, especially for $|\rho_n|$ when $n \geq 4$. Fekete-Szegő-type inequalities, higher-order Hankel determinants, and the effects of fractional calculus operators on these subclasses are some potential avenues for future study.

Further developments and wider applications in geometric function theory and related topics may result from expanding the framework created in this work by adding additional families of orthogonal polynomials or by employing q -analogues.

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