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ON THE THEORY OF HESITANT FUZZY FUNCTIONS AND THEIR REDUCTION TO FUZZY AND INTUITIONISTIC FUZZY FUNCTION MODELS

(*Teori Fungsi Kabur Teragak-agak dan Penurunannya kepada Model Fungsi Kabur dan Fungsi Kabur Berintuisi*)

MOHAMMAD TALAFHA, SYAHIDA CHE DZUL-KIFLI*, ABD ULAZEEZ ALKOURI & EMAN HUSSEIN

ABSTRACT

This paper develops a formal theoretical framework for hesitant fuzzy functions (HFFs) within the setting of hesitant fuzzy relations. Although hesitant fuzzy relations have been introduced in the literature, a corresponding function-based framework has not yet been established in a clear and systematic way. To address this gap, HFFs are defined as a special class of hesitant fuzzy relations satisfying suitable conditions that preserve consistency in hesitant fuzzy mappings. Within this framework, an HFF from X to Y is interpreted as a mapping between hesitant fuzzy sets on X and Y, and several fundamental properties are studied, including equality, injectivity, surjectivity, composition, invertibility, and image and preimage behavior. Illustrative examples and counterexamples are also provided to clarify these notions in the hesitant fuzzy setting. In addition, several basic theorems are established to support the structural behavior of HFFs. The paper further shows that the proposed framework admits natural reductions to fuzzy functions (FFs) and intuitionistic fuzzy functions (IFFs) under suitable structural constraints on the co-membership functions of HFFs. This reduction highlights the unifying role of HFFs as a general framework for representing hesitation in functional settings and provides a basis for further developments in hesitant fuzzy mappings and related structures. This work not only deepens the understanding of hesitant fuzzy mappings but also bridges the gap between different fuzzy function models, with potential applications in decision-making, pattern recognition, and artificial intelligence systems dealing with vague or imprecise information.

Keywords: fuzzy function; intuitionistic fuzzy function; hesitant fuzzy set; hesitant fuzzy Cartesian product; hesitant fuzzy relation; hesitant fuzzy function

ABSTRAK

Makalah ini membina suatu kerangka kerja teoritical yang formal bagi fungsi kabur teragak-agak (FKTA) dalam konteks hubungan kabur teragak-agak. Walaupun hubungan kabur teragak-agak telah diperkenalkan dalam kajian kepustakaan, kerangka kerja berasaskan fungsi yang sepadan masih belum dibangunkan secara jelas dan sistematik. Bagi menangani jurang ini, FKTA ditakrifkan sebagai satu kelas khas hubungan kabur teragak-agak yang memenuhi syarat tertentu bagi mengekalkan keseragaman dalam pemetaan kabur teragak-agak. Dalam kerangka kerja ini, FKTA daripada X kepada Y ditafsirkan sebagai suatu pemetaan antara set kabur teragak-agak pada X dan Y, dan beberapa sifat asas dikaji, termasuk kesamaan, sifat satu ke satu, keseluruh, gubahan, songsangan, serta tingkah laku imej dan praimej. Contoh ilustrasi dan contoh penyangkal turut disediakan bagi memperjelaskan konsep ini dalam persekitaran kabur teragak-agak. Selain itu, beberapa teorem asas dibuktikan bagi menyokong sifat struktur FKTA. Makalah ini juga menunjukkan bahawa kerangka kerja yang dicadangkan membenarkan penurunan secara semula jadi kepada fungsi kabur (FK) dan fungsi kabur berintuisi (FKB) di bawah kekangan struktur yang sesuai pada fungsi keahlian bersama FKTA. Penurunan ini memperlihatkan peranan FKTA sebagai satu kerangka kerja umum yang menyatukan pelbagai model fungsi kabur serta menyediakan asas bagi perkembangan lanjut dalam pemetaan kabur teragak-agak dan struktur berkaitan. Kajian ini bukan sahaja memperdalam pemahaman terhadap pemetaan kabur teragak-agak, malah turut merapatkan jurang antara model fungsi kabur yang berbeza, dengan potensi aplikasi dalam pembuatan keputusan, pengecaman corak, dan sistem kecerdasan buatan yang melibatkan maklumat yang kabur atau tidak tepat.

Kata kunci: fungsi kabur; fungsi kabur berintuisi; set kabur teragak-agak; hasil darab Cartesian kabur teragak-agak; hubungan kabur teragak-agak; fungsi kabur teragak-agak

1. Introduction

Many real-world problems in decision-making, pattern recognition, and artificial intelligence involve uncertainty. To model such situations, the concept of fuzzy sets (FSs) was introduced by Zadeh in 1965, where every element of a universe receives a single membership degree (Zadeh 1965). This framework was later extended by Atanassov through intuitionistic fuzzy sets (IFSs), where every element is characterized by both a membership degree and a non-membership degree satisfying $\mu + v \leq 1$ (Atanassov 1986). The remaining part, $1 - \mu - v$, represents a degree of indeterminacy, allowing IFSs to describe incomplete or ambiguous information more effectively than classical fuzzy sets.

Despite these developments, there are situations in which a single membership value, or even a single pair of membership and non-membership degrees, is insufficient to represent the opinions of the experts and the outputs of a system. In such cases, hesitation may be better expressed by several possible membership values rather than a single fixed value. To address this issue, Torra and Narukawa introduced hesitant fuzzy sets (HFSs), in which every element is linked to a finite set of possible membership degrees (Torra 2010; Torra and Narukawa 2009). HFSs may be regarded as an extension of fuzzy sets and intuitionistic fuzzy sets for situations in which hesitation cannot be adequately represented by a single value or a single pair of values. Because of their flexibility in representing hesitation, HFSs have been widely used in many areas, including multiple-criteria decision making, pattern recognition, clustering, classification, medical diagnosis, social network analysis, and information fusion (see, e.g., Alkouri *et al.* 2023; Alsharo *et al.* 2024; Alshehri *et al.* 2025; Chaube *et al.* 2025; Gao 2025; Gupta & Kumar 2024; Talafha *et al.* 2021; Wang *et al.* 2025).

As fuzzy set theory developed, it became necessary to extend classical relations and functions to fuzzy environments. In this direction, Dib and Youssef (1991) introduced a lattice-theoretic formulation of the fuzzy Cartesian product (FCP), thereby providing a consistent basis for defining fuzzy relations as subsets of the FCP. Within this framework, fuzzy functions (FFs) were defined as special fuzzy relations satisfying an appropriate functional condition. This construction established a coherent setting for studying function-related notions such as composition, identity, and inverse in fuzzy settings. Later, Fathi and Salleh (2008) extended this approach to the intuitionistic fuzzy setting by defining intuitionistic fuzzy functions (IFFs) as special intuitionistic fuzzy relations derived from the intuitionistic fuzzy Cartesian product.

More recently, Talafha *et al*. (2025) introduced the hesitant fuzzy Cartesian product (HFCP) and used it to define hesitant fuzzy relations (HFRs). Their work extended the Cartesian product-based approach from fuzzy and intuitionistic fuzzy settings to the hesitant fuzzy context and showed that hesitant structures can preserve information more effectively in evaluation problems. In a related direction, Al-Zu'bi *et al.* (2024a) extended the same Cartesian product-based framework to bipolar-valued environments, further demonstrating its flexibility in modelling richer forms of uncertainty. However, although hesitant fuzzy relations have been established, the literature still lacks a formal and systematic theory of hesitant fuzzy functions.

As a result, essential function-based notions such as equality, composition, invertibility, image, and preimage remain insufficiently developed in the hesitant fuzzy setting. While Talafha *et al.* (2025) focused on the development of hesitant fuzzy relations, the present work develops a formal function-based framework for the hesitant fuzzy setting and studies the conditions and properties that characterize hesitant fuzzy functions.

This paper addresses that gap by introducing the concept of a hesitant fuzzy function (HFF) as a specific class of hesitant fuzzy relations. The proposed definition was formulated within the framework of hesitant fuzzy Cartesian products and hesitant fuzzy relations. In this setting, each input is associated with exactly one output in the hesitant fuzzy sense, while the corresponding membership information is represented by a hesitant fuzzy element rather than a single value. In this way, the proposed model extends existing fuzzy and intuitionistic fuzzy function frameworks to situations in which hesitation must be preserved explicitly.

The novelty of this work does not lie merely in transferring the notion of a function to the hesitant fuzzy setting. Rather, it lies in establishing a formal function-based structure within hesitant fuzzy relation theory and in identifying the conditions that distinguish hesitant fuzzy functions from general hesitant fuzzy relations. This makes it possible to study the basic properties of HFFs, including equality, composition, identity, and inverse, and shows how the proposed model is connected to the existing theories of fuzzy functions and intuitionistic fuzzy functions through suitable reduction results.

The paper aims to define hesitant fuzzy functions within the framework of hesitant fuzzy relations and to investigate their fundamental theoretical properties in a systematic manner. Specifically, the paper formulates hesitant fuzzy functions as a specific class of hesitant fuzzy relations, characterizes the functional conditions that distinguish them from general hesitant fuzzy relations, establishes several of their basic properties, including equality, composition, identity, and inverse, and demonstrates that the proposed framework admits natural reductions to fuzzy functions and intuitionistic fuzzy functions. Through this formal development, the paper contributes to a more complete function-theoretic foundation for hesitant fuzzy environments and provides a rigorous basis for future research on hesitant fuzzy mappings and related algebraic constructions.

Beyond the scope of the present work, the proposed framework naturally opens the door to hesitant fuzzy algebra through suitable hesitant fuzzy spaces and hesitant fuzzy binary operations. This direction is consistent with Dib's (1994) algebraic framework for fuzzy spaces and fuzzy binary operations, and is further supported by subsequent Cartesian product-based studies on fuzzy and intuitionistic fuzzy algebraic structures (see, e.g., Fathi & Salleh 2009; Al-Husban & Salleh 2016; Jaradat & Al-Husban 2022; Al-Zu'bi *et al.* 2024b, 2025).

This paper is organized as follows. Section 2 presents the preliminary ideas needed for this study. Section 3 introduces hesitant fuzzy functions and studies their essential properties, supported by illustrative examples and proofs. Section 4 discusses the reduction of hesitant fuzzy functions to fuzzy functions and intuitionistic fuzzy functions. Finally, Section 5 concludes the paper and outlines possible directions for forthcoming research.

2. Preliminaries

This section provides a brief overview of the basic concepts related to fuzzy sets (FSs), intuitionistic fuzzy sets (IFSs), and hesitant fuzzy sets (HFSs), which serve as the foundation for developing the theory of hesitant fuzzy functions.

Definition 2.1. (Zadeh 1965) Let X be a universal set and let $I = [0, 1]$. A fuzzy set (FS) A on X is a defined as

$$A = \{\langle x, \mu_A(x)\rangle | \; x \in X\}$$

where $\mu_A: X \to I$ is the membership function, and $\mu_A(x)$ is the degree of membership of x in A.

Note that I^X denotes the collection of all functions from X to $I = [0,1]$. Consequently, every fuzzy set A on X can be viewed as an element of I^X, written $A \in I^X$.

Definition 2.2. (Dib & Youssef 1991) A fuzzy function (FF) **F** from an ordinary set X to an ordinary set Y is a fuzzy relation from X to Y satisfying the following conditions:

1) For every $x \in X$ and $r \in [0,1] = I$, there exists a unique element $y \in Y$ and $t \in I$ such that $(x, y, r, t) \in A$ for some $A \in \mathbf{F}$ and $\mu_A(x) = r$ and $\mu_A(y) = t$.
2) If $(x, y, r_1, t_1) \in A \in \mathbf{F}$ and $(x, y', r_2, t_2) \in B \in \mathbf{F}$, then $y = y'$.
3) If $(x, y, r_1, t_1) \in A \in \mathbf{F}$ and $(x, y, r_2, t_2) \in B \in \mathbf{F}$, with $r_1 > r_2$, then $t_1 \geq t_2$.
4) If $(x, y, r_1, t_1) \in A \in \mathbf{F}$, then $r = 0$ implies $t = 0$ and $r = 1$ implies $t = 1$.

Building on fuzzy sets (FSs), Atanassov in 1986 (Atanassov 1986) proposed intuitionistic fuzzy sets (IFSs) by incorporating both membership and non-membership degrees.

Definition 2.3. (Atanassov 1986) Let X be a universal set and let $I = [0, 1]$. An intuitionistic fuzzy set (IFS) B on X is given by

$$B = \{\langle x, \mu_B(x), v_B(x)\rangle : x \in X\},$$

where $\mu_B, v_B: X \longrightarrow [0,1]$ are the membership and non-membership functions of B, respectively, and satisfy the condition

$$0 \leq \mu_A(x) + v_A(x) \leq 1 \qquad \text{for all } x \in X.$$

Since $(I \times I)^X$ is the set of all functions from X to $I \times I$, every intuitionistic fuzzy set A can be viewed as an element of $(I \times I)^X$.

Definition 2.4. (Fathi & Salleh 2008) An intuitionistic fuzzy function (IFF) **F** from X to Y is an intuitionistic fuzzy relation from X to Y satisfying the following conditions:

1) For every $x \in X$ with $r, s \in [0,1] = I$, there exists a unique element $y \in Y$ and $w, z \in I$ such that $\big((x, y), (r, w), (s, z)\big) \in A$ for some $A \in \mathbf{F}$.
2) If $\big((x, y), (r_1, w_1), (s_1, z_1)\big) \in A \in \mathbf{F}$ and $\big((x, y'), (r_2, w_2), (s_2, z_2)\big) \in B \in \mathbf{F}$, then $y = y'$.
3) If $\big((x, y), (r_1, w_1), (s_1, z_1)\big) \in A \in \mathbf{F}$ and $\big((x, y), (r_2, w_2), (s_2, z_2)\big) \in B \in \mathbf{F}$ with $r_1 > r_2$, then $w_1 \geq w_2$ and if $s_1 > s_2$, then $z_1 \geq z_2$.
4) Boundary conditions: If $\big((x, y), (r, w), (s, z)\big) \in A \in \mathbf{F}$, with $r = 0$, then $w = 0$, if $s = 0$ then $z = 0$, if $r = 1$, then $w = 1$, and if $s = 1$ then $z = 1$.

The following definitions explain the main structure of hesitant fuzzy sets (HFSs), which allow multiple membership degrees to express uncertainty more precisely in fuzzy environments.

Definition 2.5. (Torra 2010; Torra & Narukawa 2009) Let X be a universal set and let $I = [0, 1]$. A hesitant fuzzy set (HFS) A on X is defined as

$$A = \{\langle x, h_A(x)\rangle | x \in X\},$$

where $h_A(x)$ is a finite subset of the closed unit interval $I = [0,1]$ representing possible membership degrees of the element x in the set A. The set $h_A(x) = \{\gamma_1, \gamma_2, \dots, \gamma_n\} \subset \mathrm{P}(I)$, for some positive integer n, refers to a hesitant fuzzy element (HFE) associated with A. The function

$$h_A : X \longrightarrow P(I)$$

is called the hesitant fuzzy membership function (HFMF) of A.

Sometimes, we omit the notation A and simply write $h(x)$ instead of $h_A(x)$ to indicate that $h(x)$ is an HFE associated with some HFS on X. We also write $h_A \in P(I)^X$ where $P(I)^X$ indicates the set of all functions from X to $P(I)$. Since $P(I)^X$ is the collection of all functions from X to $P(I)$, so we also write any hesitant fuzzy set A on X as $A \in P(I)^X$. Since the image $h_A(x) \subset P(I)$ is finite, we sometimes use the set $P(I)^X_{\text{finite}}$ as the range of $h_A(x)$ for any $A \in P(I)^X$.

Example 2.1. (Torra 2010; Torra & Narukawa 2009) Let X be a universal set. The following are specific forms of hesitant fuzzy sets on X:

1) Empty HFS: $O^* = \{\langle x, \{0\}\rangle |\ x \in X\} = \{\ \}$ with $h(x) = \{0\}$.
2) Full HFS: $I^* = \{\langle x, \{1\}\rangle |\ x \in X\}$ with $h(x) = \{1\}$.
3) Complete ignorance HFS: $U^* = \{\langle x, [0,1]\rangle | x \in X\}$ i.e. whenever $h(x) = [0,1]$ for all x.
4) Nonsense (undefined) HFS: $\emptyset^* = \{\langle x, \emptyset\rangle |\ x \in X\}$ i.e. whenever $h(x) = \emptyset$ for all x.

For (2) the ordinary set X is considered an HFS of itself. Now we consider operations and relations on these sets to build the structure of hesitant fuzzy set theory.

Definition 2.6. (Torra 2010) Let A and B be two HFSs defined on a universal set X, with corresponding HFMF h_A and h_B maps from X to $P(I)$, where $I = [0, 1]$. The union and intersection of A and B, denoted by $A \cup B$ and $A \cap B$ respectively, are also HFSs on X. Their HFMF, denoted by $h_{A\cup B}$ and $h_{A\cap B}$ are defined as follows

Union: $$h_{A\cup B}(x) = \{\delta \in h_A(x) \cup h_B(x) | \delta \geq \max\{h_A^-(x), h_B^-(x)\} \mid x \in X\},$$

Intersection: $$h_{A\cap B}(x) = \{\delta \in h_A(x) \cup h_B(x) | \delta \leq \min\{h_A^+(x), h_B^+(x)\} \mid x \in X\},$$

where the lower bound and upper bounds of HFE $h_A(x)$ are defined respectively as:

$$h_A^-(x) = \min\{h_A(x)\} \quad \text{and} \quad h_A^+(x) = \max\{h_A(x)\}$$

and similarly, for $h_B(x)$.

Definition 2.7. (Talafha *et al.* 2025) Let X be a universal set, $A = \{\langle x, h_A(x)\rangle | x \in X\}$ and $B = \{\langle x, h_B(x)\rangle | x \in X\}$ be two HFSs on X. Then we say that A is a subset of B (denoted as $A \subseteq B$), iff for every $x \in X$,

$$mean\big(h_A(x)\big) \le mean\big(h_B(x)\big)$$

where

$$mean\big(h_A(x)\big) = \begin{cases} \frac{\sum_{i=1}^{n} \gamma_i}{n} & \text{if} \quad h_A = \{\gamma_1, \gamma_2, \dots, \gamma_n\} \\ 1 & \text{if} \quad h_A = [0,1] \\ 0 & \text{if} \quad h_A = \emptyset \end{cases}.$$

Moreover, A is a proper subset of B, denoted as $A \subset B$, if

$$mean\ \big(h_A(x)\big) \le mean\ \big(h_B(x)\big) \text{ for all } x \in X,$$

and

$$mean\ \big(h_A(x)\big) < mean\ \big(h_B(x)\big) \text{ for some } x \in X.$$

Definition 2.8. (Talafha *et al.* 2025) Let $A = \{\langle x, h_A(x)\rangle | x \in X\}$ and $B = \{\langle x, h_B(x)\rangle | x \in X\}$ be two HFSs on X. Then we say that A and B are equal (denoted as $A = B$), iff for every $x \in X$,

$$mean\big(h_A(x)\big) = mean\big(h_B(x)\big),$$

where $h_A(x)$ and $h_B(x)$ are HFEs of A and B respectively.

Motivated by Dib's work on fuzzy sets, we extend classical operations and relations into a richer structure by introducing M-hesitant fuzzy sets (M-HFSs) on Cartesian products. Using $M = P(I) \times P(I)$ where $I = [0,1]$, an M-HFS assigns to each $(x, y) \in X \times Y$ a pair of hesitant membership degrees, enabling the definition of the hesitant fuzzy Cartesian product (HFCP) and its role in modelling hesitant fuzzy relations (HFRs). Accordingly, we define the following;

Definition 2.9. For two HFSs $A, B \in P(I)^X$ on X, hesitant fuzzy membership function (HFMF) of A and B is equal (denoted as $h_A(x) = h_B(x)$ for all $x \in X$) whenever

$$mean\big(h_A(x)\big) = mean\big(h_B(x)\big).$$

Definition 2.10. (Torra 2010; Torra & Narukawa 2009) Let X be a universal set, and A be an HFS on X. The envelope of the HFE of A is defined as:

$$E_{env}(A) = \{\langle x, \{\mu_A(x), v_A(x)\}\rangle | x \in X\}$$

where $\mu_A(x) = h_A^-(x)$ and $v_A(x) = 1 - h_A^+(x)$.

Definition 2.11. (Talafha *et al.* 2025) Let X and Y be universal sets, let $I = [0,1]$ and $M = P(I) \times P(I)$. An M-hesitant fuzzy subset (M-HFS) on $X \times Y$ is a function that assigns to each $(x, y) \in X \times Y$ an element of M. To be easily understood, we express the M-HFS (say the set A^M) on $X \times Y$ by a mathematical symbol:

$$A^M = \Big\{\langle (x, y), \Big(h_A^1(x, y), h_A^2(x, y)\Big)\rangle \mid (x, y) \in X \times Y\Big\},$$

where $h_A^1(x,y), h_A^2(x,y)$ are finite subsets of [0,1]. $h_{A^M}(x,y) = \left(h_A^1(x,y), h_A^2(x,y)\right)$ is called an M-HFE of A^M on $X \times Y$.

Definition 2.12. (Talafha *et al.* 2025) The hesitant fuzzy Cartesian product (HFCP) of two ordinary sets X and Y, denoted by $X \ddot{\times} Y$, is the collection of all *M*-hesitant fuzzy subsets on $X \times Y$.

Consequently, $X \ddot{\times} Y = M^{X\times Y}$ i.e., a collection of all functions from $X \times Y$ to $M = P(I) \times P(I)$. An element of $X \ddot{\times} Y$ is then an M-HFS on $X \times Y$,

$$A^M = \left\{\langle (x,y), \left(h_A^1(x,y), h_A^2(x,y)\right)\rangle \mid (x,y) \in X \times Y\right\} \in X \ddot{\times} Y.$$

Definition 2.13. (Talafha *et al.* 2025) For two HFSs $A = \{\langle x, h_A(x)\rangle | x \in X\}$ and $B = \{\langle y, h_B(y)\rangle | y \in Y\}$ on X and Y respectively, the hesitant fuzzy Cartesian product (HFCP) of A and B (denoted by $A \dot{\times} B$) is the *M*-HFS on $X \times Y$ generated by A and B as follows.

$$A \dot{\times} B = \left\{\langle (x,y), \left(h_A(x), h_B(y)\right)\rangle | x \in X, y \in Y\right\},$$

which is an *M*-hesitant fuzzy subset of $X \times Y$.

Consequently, $A \dot{\times} B$ is an element of $X \ddot{\times} Y$. Therefore, we may characterize any element of $X \ddot{\times} Y$ as either an M-hesitant fuzzy subset A^M on $X \times Y$ or as a hesitant fuzzy Cartesian product $A \dot{\times} B$ of two HFSs A and B on X and Y, respectively.

Lemma 2.1. (Talafha *et al.* 2025) The HFCP of two ordinary sets X and Y, $X \ddot{\times} Y$, is the collection of all HFCPs of A and B, $A \dot{\times} B$ for any two HFSs A and B on X and Y, respectively.

Definition 2.14. (Talafha *et al.* 2025) A hesitant fuzzy relation (HFR) **R** from X to Y is a subset of the HFCP $X \ddot{\times} Y$. If **R** is from X to X, then **R** is called a hesitant fuzzy relation on X.

3. Hesitant Fuzzy Function and its Essential Properties

In this section, we introduce the concept of hesitant fuzzy functions (HFFs), grounded in set theory and hesitant fuzzy relations. We provide a rigorous definition of HFFs based on four fundamental conditions that characterize their behavior. Building on this definition, we examine the essential properties of HFFs, including composability, nondecreasing behavior, image uniqueness, and injectivity. To support the theoretical framework, we include several detailed examples and counterexamples that demonstrate how these properties manifest in practice.

3.1. *Hesitant Fuzzy Function*

Let us recall that a hesitant fuzzy relation **R** from an ordinary set X to an ordinary set Y is any subset of the hesitant fuzzy Cartesian product $X \ddot{\times} Y$. According to Lemma 2.1, we may regard the element of **R** as an M-hesitant fuzzy set given by

$$A^M = \left\{\langle (x,y), \left(h_A^1(x,y), h_A^2(x,y)\right)\rangle | (x,y) \in X \times Y\right\}$$

which is defined on the Cartesian product $X \times Y$, or more specifically, as a hesitant fuzzy Cartesian product $A \dot{\times} B$ of two hesitant fuzzy sets A and B on X and Y, respectively. That is,

$$A \dot{\times} B = \{\langle (x, y), (h_A(x), h_B(y)) \rangle | x \in X, y \in Y\},$$

where $h_A \in P(I)^X$ and $h_B \in P(I)^Y$ both have finite images.

This section interprets each element of any hesitant fuzzy relation as a hesitant fuzzy Cartesian product $\mathcal{A} = A \dot{\times} B$ of two hesitant fuzzy sets A and B on X and Y, respectively. For simplicity of notation, we may occasionally write

$$\mathcal{A} = \{\langle (x, y), (h(x), h(y)) \rangle | x \in X, y \in Y\}$$

and omit explicit reference to the associated hesitant fuzzy sets A and B. Thus, we present our hesitant fuzzy function as a hesitant fuzzy relation with some conditions as stated in the following definition;

Definition 3.1.1. A hesitant fuzzy function (HFF) from an ordinary set X to another ordinary set Y is defined as a hesitant fuzzy relation $\mathbf{F} \subseteq X \ddot{\times} Y$ that satisfies the following conditions:

1. Existence: For every HFS A on X, there exists an HFS B on Y, such that $A \dot{\times} B \in \mathbf{F}$.
2. Uniqueness: If $A \dot{\times} B \in \mathbf{F}$ and $A \dot{\times} D \in \mathbf{F}$, then $B = D$.
3. Order preserving (nondecreasing property): Let $A \dot{\times} B \in \mathbf{F}$ and $C \dot{\times} D \in \mathbf{F}$, for HFSs A and C on X, and HFSs B and D on Y. Then for any $x \in X$, if

   $$mean\,(h_A(x)) \leq mean\,(h_C(x))$$

   then

   $$mean\,(h_B(y)) \leq mean\,(h_D(y)).$$

4. Boundary preservation: For every $\langle (x, y), (h(x), h(y)) \rangle \in \mathcal{A} \in \mathbf{F}$, if $h(x) = \{0\}$, then $h(y) = \{0\}$ and if $h(x) = \{1\}$, then $h(y) = \{1\}$.

The nondecreasing condition is necessary in the definition of hesitant fuzzy functions because it ensures that the order of uncertainty or preference is preserved when mapping one hesitant fuzzy set to another. This is particularly important since, unlike classical functions where each element has a single membership value, hesitant fuzzy elements contain multiple membership values. Without this condition, a set with higher membership values could be mapped to lower ones, which would contradict logical reasoning in applications such as decision-making or ranking. Thus, this condition helps maintain consistency and meaningful interpretation of hesitant fuzzy mappings.

The boundary preservation condition is also essential in hesitant fuzzy functions to ensure that the extreme values of uncertainty are maintained during the mapping. Specifically, if the input has a membership value of 0 (complete non-membership), the output should also be 0, and if the input is 1 (full membership), the output should be 1. This mirrors the classical idea that definite inputs should lead to definite outputs, preserving semantic meaning and maintaining logical consistency at the boundaries of the membership scale.

Example 3.1.1. Let $X = \{1, 2, 3,4,5,6\}$ and $Y = \mathbb{Z}$ be universal sets, and let

$$\mathbf{R} = \left\{ \begin{array}{c} \left\{ \begin{array}{c} \langle(1,2), (\{0.4, 0.5, 0.7\}, \{0.16, 0.25, 0.49\})\rangle, \langle(1,4), (\{0.4, 0.5, 0.7\}, \{0.16, 0.49, 0.36\})\rangle, \\ \langle(1,6), (\{0.4, 0.5, 0.7\}, \{0.25, 0.36, 0.64\})\rangle, \langle(2,2), (\{0.4, 0.7, 0.6\}, \{0.16, 0.25, 0.49\})\rangle, \\ \langle(2,4), (\{0.4, 0.7, 0.6\}, \{0.16, 0.49, 0.36\})\rangle, \langle(2,6), (\{0.4, 0.7, 0.6\}, \{0.25, 0.36, 0.64\})\rangle, \\ \langle(3,2), (\{0.5, 0.6, 0.8\}, \{0.16, 0.25, 0.49\})\rangle, \langle(3,4), (\{0.5, 0.6, 0.8\}, \{0.16, 0.49, 0.36\})\rangle, \\ \langle(3,6), (\{0.5, 0.6, 0.8\}, \{0.25, 0.36, 0.64\})\rangle \end{array} \right\}, \\ \left\{ \begin{array}{c} \langle(1, 2), (\{0.1, 0.3\}, \{0.01, 0.09\})\rangle, \langle(1, 4), (\{0.1, 0.3\}, \{0.25, 0.01, 0.36\})\rangle \\ \langle(1, 6), (\{0.1, 0.3\}, \{0.16, 0.36, 0.64\})\rangle, \langle(2, 2), (\{0.5, 0.1, 0.6\}, \{0.01, 0.09\})\rangle \\ \langle(2, 4), (\{0.5, 0.1, 0.6\}, \{0.25, 0.01, 0.36\})\rangle, \langle(2, 6), (\{0.5, 0.1, 0.6\}, \{0.16, 0.36, 0.64\})\rangle \\ \langle(3, 2), (\{0.4, 0.6, 0.8\}, \{0.01, 0.09\})\rangle, \langle(3, 4), (\{0.4, 0.6, 0.8\}, \{0.25, 0.01, 0.36\})\rangle \\ \langle(3, 6), (\{0.4, 0.6, 0.8\}, \{0.16, 0.36, 0.64\})\rangle \end{array} \right\}, \\ \left\{ \begin{array}{c} \langle(4, 2), (\{0.4, 0.5, 0.7\}, \{0.16, 0.25, 0.49\})\rangle, \langle(4, 4), (\{0.4, 0.5, 0.7\}, \{0.16, 0.49, 0.36\})\rangle, \\ \langle(4, 6), (\{0.4, 0.5, 0.7\}, \{0.25, 0.36, 0.64\})\rangle, \langle(5, 2), (\{0.4, 0.7, 0.6\}, \{0.16, 0.25, 0.49\})\rangle, \\ \langle(5, 4), (\{0.4, 0.7, 0.6\}, \{0.16, 0.49, 0.36\})\rangle, \langle(5, 6), (\{0.4, 0.7, 0.6\}, \{0.25, 0.36, 0.64\})\rangle, \\ \langle(6, 2), (\{0.5, 0.6, 0.8\}, \{0.16, 0.25, 0.49\})\rangle, \langle(6, 4), (\{0.5, 0.6, 0.8\}, \{0.16, 0.49, 0.36\})\rangle, \\ \langle(6, 6), (\{0.5, 0.6, 0.8\}, \{0.25, 0.36, 0.64\})\rangle \end{array} \right\} \end{array} \right\}$$

be a hesitant fuzzy relation from X to Y. Every element in the set satisfies the existence condition. The uniqueness condition is satisfied since each domain HFS corresponds to only one codomain HFS. The nondecreasing condition holds as the mean values of $h(x)$ and $h(y)$ preserve the ordering. The boundary values (0 and 1) are preserved. Therefore, the hesitant fuzzy relation **R** is a hesitant fuzzy function **F** from X to Y.

As mentioned earlier, the element of a hesitant fuzzy relation can be written as a hesitant fuzzy Cartesian product $A \dot{\times} B$ of two hesitant fuzzy sets A and B on X and Y, respectively. Therefore, according to Definition 2.13, the HFCP $A \dot{\times} B$ of two HFSs A and B on X and Y respectively, is generated by A and B, and the converse is also true. Therefore, from the first element of the hesitant fuzzy relation (say $\mathcal{A}$) we can derive hesitant fuzzy sets A and B on X and Y such that $\mathcal{A} = A \dot{\times} B$ represents the first element from the relation **R** in Example 3.1.1, given as follows:

$A = \{\langle 1, \{0.4, 0.5, 0.7\}\rangle, \langle 2, \{0.4, 0.7, 0.6\}\rangle,\ \langle 3, \{0.5, 0.6, 0.8\}\rangle\}$ be an HFS on X, and

$B = \{\langle 2, \{0.16, 0.25, 0.49\}\rangle, \langle 4, \{0.16, 0.49, 0.36\}\rangle, \langle 6, \{0.25, 0.36, 0.64\}\rangle\}$ is an HFS on Y.

Similarly, from the second element of the hesitant fuzzy relation (say $\mathcal{B}$) we may write it as $C \dot{\times} D$, where:

$C = \{\langle 1, \{0.1, 0.3\}\rangle, \langle 2, \{0.5, 0.1, 0.6\}\rangle, \langle 3, \{0.4, 0.6, 0.8\}\rangle\}$ is an HFS on X and

$D = \{\langle 2, \{0.01, 0.09\}\rangle, \langle 4, \{0.25, 0.01, 0.36\}\rangle, \langle 6, \{0.16, 0.36, 0.64\}\rangle\}$ is an HFS on Y.

And, from the third element of the hesitant fuzzy relation (say $\mathcal{C}$), we may write it as $E \dot{\times} F$ where,

$E = \{\langle 4, \{0.4, 0.5, 0.7\}\rangle, \langle 5, \{0.4, 0.7, 0.6\}\rangle,\ \langle 6, \{0.5, 0.6, 0.8\}\rangle\}$, be an HFS on X, and

$F = \{\langle 2, \{0.16, 0.25, 0.49\}\rangle, \langle 4, \{0.16, 0.49, 0.36\}\rangle, \langle 6, \{0.25, 0.36, 0.64\}\rangle\} = B$, is an HFS on Y.

Therefore

$$\mathbf{R} = \{\mathcal{A}, \mathcal{B}, \mathcal{C}\} = \{A \dot{\times} B, C \dot{\times} D, E \dot{\times} B\}.$$

Conditions 1 and 2 of Definition 3.1.1 are obviously satisfied by **R**.

For condition 3, if we consider $\mathcal{A} = A \dot{\times} B \in \zeta$, $x = 1 \in X$, $y = 2 \in Y$, then $mean(h_{\mathcal{A}}(1)) = 0.5\overline{3}$, $mean(h_{\mathcal{A}}(2)) = 0.3$, $mean(h_{\mathcal{B}}(1)) = 0.2$, $mean(h_{\mathcal{B}}(2)) = 0.05$. So $mean(h_{\mathcal{B}}(1)) \leq mean(h_{\mathcal{A}}(1))$ and $mean(h_{\mathcal{B}}(2)) \leq mean(h_{\mathcal{A}}(2))$ which means that these selections of elements satisfy Condition 3 of Definition 3.1.1. The other values of $x \in X$, $y \in Y$, for all $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are also true in the same way of proving.

For condition 4, we observe that $h(x) \neq \{0\}$ and $h(\mathrm{x}) \neq \{1\}$ for all $x \in X$; hence the boundary cases do not occur, and the boundary preservation condition is trivially satisfied. Therefore, the relation **R** is a hesitant fuzzy function from X to Y, and we denote this function by **F**.

Condition 1 (existence) and condition 2 (uniqueness) in Definition 3.1.1 imply that for every hesitant fuzzy function $\mathbf{F}: P(I)^X \longrightarrow P(I)^Y$, we can find a unique ordinary function $\mathcal{F}: X \longrightarrow Y$ such that $\mathcal{F}(x) = y$ iff $\langle (x, y), (h(x), h(y))\rangle \in \mathcal{A}$ for some $\mathcal{A} \in \mathbf{F}$. Indeed, for each $x \in X$, the pair (x, y) appears in the relation with a unique y determined by the pair $(h(x), h(y))$, hence defining a unique $\mathcal{F}: X \to Y$.

Moreover, for every $x \in X$ there exists an ordinary "co-membership" function $f_x: P(I) \longrightarrow P(I)$ which maps $U \in P(I)$ to other $V \in P(I)$ such that

$$f_x(U) = V = \begin{cases} h(y) & if \quad U = h(x) \\ \emptyset & if \quad U \neq h(x) \end{cases}$$

whenever $\langle (x, y), (h(x), h(y))\rangle \in \mathcal{A}$ for some $\mathcal{A} \in \mathbf{F}$. The function f_x, maps a hesitant fuzzy element U to $h(y)$ only when $U = h(x)$, ensuring a well-defined correspondence between input and output membership values. This satisfies the definition of a co-membership function, as it preserves the unique pairing in $\langle (x, y), (h(x), h(y))\rangle \in \mathcal{A}$ for some $\mathcal{A} \in \mathbf{F}$.

Remark 3.1.1. We now use the notation $\mathbf{F} = \{\langle \mathcal{F}, f_x\rangle : x \in X\}$ to represent a hesitant fuzzy function from X to Y. The functions f_x, for all $x \in X$, are called the co-membership function associated with **F**, as motivated by the analog term used by (Dib & Youssef 1991). Therefore, we may write the image of a hesitant fuzzy set A on X under **F** as

$$\begin{aligned} \mathbf{F}(A) &= \{\langle \mathcal{F}(x), f_x(h_A(x))\rangle \mid x \in X\} \\ &= \{\langle y, h_B(y)\rangle \mid y \in Y\} \\ &= B \end{aligned}$$

where B is a hesitant fuzzy set on Y and $A \dot{\times} B \in \mathbf{F}$. Now, since A and B are hesitant fuzzy sets on X and Y respectively, therefore, we may now use the notation $\mathbf{F} = \{\langle \mathcal{F}, f_x\rangle : x \in X\}$ as $\mathbf{F}: P(I)^X \to P(I)^Y$ or simply, $\mathbf{F} = \{\langle \mathcal{F}, f_x\rangle\}: P(I)^X \to P(I)^Y$ to represent a hesitant fuzzy function **F** from X to Y. To maintain clarity, we note that $f_x(h_A(x)) = h_B(\mathcal{F}(x))$ whenever $\mathcal{F}(x) = y$.

Example 3.1.2. Let us consider a hesitant fuzzy function **F** as in Example 3.1.1. So $\mathcal{F}: X \to Y$ is a function defined as follows

$$\mathcal{F}(x) = \begin{cases} \mathcal{F}_1(x) = 2x & \text{if} \quad x = 1,2,3 \\ \mathcal{F}_2(x) = 2 & \text{if} \quad x = 4 \\ \mathcal{F}_3(x) = 4 & \text{if} \quad x = 5 \\ \mathcal{F}_4(x) = 6 & \text{if} \quad x = 6 \end{cases}.$$

The co-membership function of the hesitant fuzzy function is generated by the element $x \in X$. For $x = 1$, we have

$$f_1(U) = \begin{cases} \{0.16, 0.25, 0.49\} & \text{if} \quad U = \{0.4, 0.5, 0.7\} \\ \{0.01, 0.09\} & \text{if} \quad U = \{0.1, 0.3\} \\ \emptyset & \text{if} \quad \text{otherwise} \end{cases}$$

since $\mathcal{F}(1) = 2$, $h_A(1) = \{0.4, 0.5, 0.7\}$, $h_B(2) = \{0.16, 0.25, 0.49\}$, and $h_C(1) = \{0.1, 0.3\}$, $h_D(2) = \{0.01, 0.09\}$, $A \dot{\times} B, C \dot{\times} D \in$ **F**.

For $x = 2$, we have

$$f_2(U) = \begin{cases} \{0.16, 0.49, 0.36\} & \text{if} \quad U = \{0.4, 0.7, 0.6\} \\ \{0.25, 0.01, 0.36\} & \text{if} \quad U = \{0.5, 0.1, 0.6\} \\ \emptyset & \text{if} \quad \text{otherwise} \end{cases}$$

since $\mathcal{F}(2) = 4$, $h_A(2) = \{0.4, 0.7, 0.6\}$, $h_B(4) = \{0.16, 0.49, 0.36\}$ and $h_C(2) = \{0.5, 0.1, 0.6\}$, $h_D(4) = \{0.25, 0.01, 0.36\}$, $A \dot{\times} B, C \dot{\times} D \in$ **F**.

For $x = 3$, we have

$$f_3(U) = \begin{cases} \{0.25, 0.36, 0.64\} & \text{if} \quad U = \{0.5, 0.6, 0.8\} \\ \{0.16, 0.36, 0.64\} & \text{if} \quad U = \{0.4, 0.6, 0.8\} \\ \emptyset & \text{if} \quad \text{otherwise} \end{cases}$$

since $\mathcal{F}(3) = 6, h_A(3) = \{0.5, 0.6, 0.8\}$, $h_B(6) = \{0.25, 0.36, 0.64\}$, $h_C(2) = \{0.4, 0.6, 0.8\}$, $h_D(4) = \{0.16, 0.36, 0.64\}$, $A \dot{\times} B, C \dot{\times} D \in$ **F**.

For $x = 4$, we have

$$f_4(U) = \begin{cases} \{0.16, 0.25, 0.49\} & \text{if} \quad U = \{0.4, 0.5, 0.7\} \\ \emptyset & \text{if} \quad \text{otherwise} \end{cases}$$

Since $\mathcal{F}(4) = 2$, $h_E(4) = \{0.4, 0.5, 0.7\}$, $h_B(2) = \{0.16, 0.25, 0.49\}$, $E \dot{\times} B \in$ **F**.

For $x = 5$, we have

$$f_5(U) = \begin{cases} \{0.16, 0.49, 0.36\} & \text{if} \quad U = \{0.4, 0.7, 0.6\} \\ \emptyset & \text{if} \quad \text{otherwise} \end{cases}$$

since $\mathcal{F}(5) = 4$, $h_E(5) = \{0.4, 0.7, 0.6\}$, $h_B(4) = \{0.16, 0.49, 0.36\}$, $E \dot{\times} B \in$ **F**.

For $x = 6$, we have

$$f_6(U) = \begin{cases} \{0.25, 0.36, 0.64\} & \text{if} \quad U = \{0.5, 0.6, 0.8\} \\ \emptyset & \text{if} \quad \text{otherwise} \end{cases}$$

Since $\mathcal{F}(6) = 4$, $h_E(6) = \{0.5, 0.6, 0.8\}$, $h_B(6) = \{0.25, 0.36, 0.64\}$, $E \dot{\times} B \in \mathbf{F}$.

From Remark 3.1.1, we found that $\mathbf{F}(A) = B, \mathbf{F}(C) = D$ and $\mathbf{F}(E) = B$, confirming that the hesitant fuzzy function $\mathbf{F}$ properly maps hesitant fuzzy sets via the co-membership function f_x for all $x \in X$. Therefore, every hesitant fuzzy function can be represented or interpreted more comprehensively (classical), as given by the following lemma.

Lemma 3.1.1. Let X and Y be nonempty sets and $\mathbf{F}: P(I)^X \to P(I)^Y$ be a hesitant fuzzy function from X to Y. Then $\mathbf{F}(A) = B$ if and only if $A \dot{\times} B \in \mathbf{F}$.

Proof. Firstly, let $\mathbf{F}: P(I)^X \to P(I)^Y$ be a hesitant fuzzy function from X to Y, where $\mathbf{F} = \{\langle \mathcal{F}(x), f_x(h(x)) \rangle | x \in X\}$ as explained in Remark 3.1.1. Let $\mathcal{F}: X \to Y$ be the underlying ordinary function defined by $\mathcal{F}(x) = y$ whenever $\langle (x, y), (h(x), h(y)) \rangle \in \mathcal{A} \in \mathbf{F}$, for each $x \in X$, let $f_x: P(I) \to P(I)$ be given by

$$f_x(U) = \begin{cases} h(y) & if \quad U = h(x) \\ \emptyset & if \quad U \neq h(x) \end{cases}$$

whenever $\langle (x, y), (h(x), h(y)) \rangle \in \mathcal{A} \in \mathbf{F}$.

Now let $A = \{\langle x, h_A(x) \rangle : x \in X\}$ be an HFS on X. by the definition of $\mathbf{F}$, we have

$$\begin{aligned} \mathbf{F}(A) &= \mathbf{F}(\{\langle x, h_A(x) \rangle : x \in X\}) \\ &= \{\langle \mathcal{F}(x), f_x(h_A(x)) \rangle : x \in X\}, \\ &= \{\langle y, h_B(y) \rangle : y \in Y\} = B, \end{aligned}$$

which is an HFS on Y. Therefore $\mathbf{F}(A) = B$.

Secondly, suppose that $\mathbf{F}: P(I)^X \to P(I)^Y$ is a mapping such that $\mathbf{F}(A) = B$ whenever $A \dot{\times} B \in \mathbf{F}$. For an HFS $A = \{\langle x, h_A(x) \rangle : x \in X\}$ on X,

$$\begin{aligned} \mathbf{F}(A) &= \mathbf{F}(\{\langle x, h_A(x) \rangle : x \in X\}) \\ &= \{\langle \mathcal{F}(x), f_x(h_A(x)) \rangle : x \in X\} = \{\langle y, h_B(y) \rangle : y \in Y\}, \end{aligned}$$

so $\mathcal{F}(x) = y$ and $f_x(h_A(x)) = h_B(y)$. Therefore, $\mathcal{F}: X \to Y$ and $f_x: P(I) \to P(I), \forall x \in X$ where $\mathbf{F} = \{\langle \mathcal{F}, f_x \rangle | x \in X\}$ is a hesitant fuzzy function from X to Y. □

Example 3.1.3. Now observe that from Example 3.1.1 for $A = \{\langle 1, \{0.4, 0.5, 0.7\} \rangle, \langle 2, \{0.4, 0.7, 0.6\} \rangle, \langle 3, \{0.5, 0.6, 0.8\} \rangle\}$ which is an HFS on X,

$$\begin{aligned} \mathbf{F}(A) &= \{\langle \mathcal{F}(1), f_1(\{0.4, 0.5, 0.7\}) \rangle, \langle \mathcal{F}(2), f_2(\{0.4, 0.7, 0.6\}) \rangle, \langle \mathcal{F}(3), f_3(\{0.5, 0.6, 0.8\}) \rangle\} \\ &= \{\langle 2, \{0.16, 0.25, 0.49\} \rangle, \langle 4, \{0.16, 0.49, 0.36\} \rangle, \langle 6, \{0.25, 0.36, 0.64\} \rangle\} = B, \end{aligned}$$

which is an HFS on Y. Then the hesitant fuzzy Cartesian product of A and B as follows:

$$\mathcal{A} = A \dot{\times} B$$

$$= \left\{ \begin{array}{c} \langle(1,2), (\{0.4, 0.5, 0.7\}, \{0.16, 0.25, 0.49\})\rangle, \langle(1,4), (\{0.4, 0.5, 0.7\}, \{0.16, 0.49, 0.36\})\rangle, \\ \langle(1,6), (\{0.4, 0.5, 0.7\}, \{0.25, 0.36, 0.64\})\rangle, \langle(2,2), (\{0.4, 0.7, 0.6\}, \{0.16, 0.25, 0.49\})\rangle, \\ \langle(2,4), (\{0.4, 0.7, 0.6\}, \{0.16, 0.49, 0.36\})\rangle, \langle(2,6), (\{0.4, 0.7, 0.6\}, \{0.25, 0.36, 0.64\})\rangle, \\ \langle(3,2), (\{0.5, 0.6, 0.8\}, \{0.16, 0.25, 0.49\})\rangle, \langle(3,4), (\{0.5, 0.6, 0.8\}, \{0.16, 0.49, 0.36\})\rangle, \\ \langle(3,6), (\{0.5, 0.6, 0.8\}, \{0.25, 0.36, 0.64\})\rangle \end{array} \right\}$$

is an element of a hesitant fuzzy function $\mathbf{F}$. Therefore, $A \dot{\times} B \in \mathbf{F}$. For

$$C = \{\langle 1, \{0.1, 0.3\}\rangle, \langle 2, \{0.5, 0.1, 0.6\}\rangle, \langle 3, \{0.4, 0.6, 0.8\}\rangle\},$$

$$\mathbf{F}(C) = \{\langle \mathcal{F}(1), f_1(\{0.1, 0.3\})\rangle, \langle \mathcal{F}(2), f_2(\{0.5, 0.1, 0.6\})\rangle, \langle \mathcal{F}(3), f_3(\{0.6, 0.6, 0.8\})\rangle\}$$

$$= \{\langle 2, \{0.01, 0.09\}\rangle, \langle 4, \{0.25, 0.01, 0.36\}\rangle, \langle 6, \{0.16, 0.36, 0.64\}\rangle\} = D,$$

which is an HFS on Y. Then the hesitant fuzzy Cartesian product of C and D as follows:

$$\mathcal{B} = C \dot{\times} D$$

$$= \left\{ \begin{array}{c} \langle(1,2), (\{0.1, 0.3\}, \{0.01, 0.09\})\rangle, \langle(1,4), (\{0.1, 0.3\}, \{0.25, 0.01, 0.36\}\ )\rangle \\ \langle(1,6), (\{0.1, 0.3\}, \{0.19, 0.36, 0.64\})\rangle, \langle(2,2), (\{0.5, 0.1, 0.6\}, \{0.01, 0.09\})\rangle \\ \langle(2,4), (\{0.5, 0.1, 0.6\}, \{0.25, 0.01, 0.36\})\rangle, \langle(2,6), (\{0.5, 0.1, 0.6\}, \{0.16, 0.36, 0.64\})\rangle \\ \langle(3,2), (\{0.6, 0.6, 0.8\}, \{0.01, 0.09\})\rangle, \langle(3,4), (\{0.6, 0.6, 0.8\}, \{0.25, 0.01, 0.36\})\rangle \\ \langle(3,6), (\{0.6, 0.6, 0.8\}, \{0.16, 0.36, 0.64\})\rangle \end{array} \right\}$$

is an element of a hesitant fuzzy function $\mathbf{F}$. Therefore, $C \dot{\times} D \in \mathbf{F}$. For

$$E = \{\langle 4, \{0.4, 0.5, 0.7\}\rangle, \langle 5, \{0.4, 0.7, 0.6\}\rangle, \langle 6, \{0.5, 0.6, 0.8\}\rangle\},$$

$$\mathbf{F}(E) = \{\langle \mathcal{F}(4), f_4(\{0.4, 0.5, 0.7\})\rangle, \langle \mathcal{F}(5), f_5(\{0.4, 0.7, 0.6\})\rangle, \langle \mathcal{F}(6), f_6(\{0.5, 0.6, 0.8\})\rangle\}$$

$$= \{\langle 2, \{0.16, 0.25, 0.49\}\rangle, \langle 4, \{0.16, 0.49, 0.36\}\rangle, \langle 6, \{0.25, 0.36, 0.64\}\rangle\} = B,$$

which is an HFS on Y. Then the hesitant fuzzy Cartesian product of E and B as follows:

$$\mathcal{C} = E \dot{\times} B$$

$$= \left\{ \begin{array}{c} \langle(4,2), (\{0.4, 0.5, 0.7\}, \{0.16, 0.25, 0.49\})\rangle, \langle(5,4), (\{0.4, 0.5, 0.7\}, \{0.16, 0.49, 0.36\})\rangle, \\ \langle(6,6), (\{0.4, 0.5, 0.7\}, \{0.25, 0.36, 0.64\})\rangle, \langle(4,2), (\{0.4, 0.7, 0.6\}, \{0.16, 0.25, 0.49\})\rangle, \\ \langle(5,4), (\{0.4, 0.7, 0.6\}, \{0.16, 0.49, 0.36\})\rangle, \langle(6,6), (\{0.4, 0.7, 0.6\}, \{0.25, 0.36, 0.64\})\rangle, \\ \langle(4,2), (\{0.5, 0.6, 0.8\}, \{0.16, 0.25, 0.49\})\rangle, \langle(5,4), (\{0.5, 0.6, 0.8\}, \{0.16, 0.49, 0.36\})\rangle, \\ \langle(6,6), (\{0.5, 0.6, 0.8\}, \{0.25, 0.36, 0.64\})\rangle \end{array} \right\}$$

is an element of a hesitant fuzzy function $\mathbf{F}$. Therefore, $E \dot{\times} B \in \mathbf{F}$.

From now on, an HFF $\mathbf{F}$ from X to Y is also a mapping from the collection of some HFSs on X to another collection of some HFSs on Y. So, for any HFSs A and B on X and Y, respectively, we have

$$\mathbf{F}(A) = B \qquad \text{whenever} \qquad A \dot{\times} B \in \mathbf{F}.$$

In addition to Remark 3.1.1, we may also write the HFF **F** as

$$\mathbf{F} = \{\langle \mathcal{F}, f_x \rangle | x \in X\}: P(I)^X \to P(I)^Y.$$

Notice that $\mathcal{F}$ in Example 3.1.2 is not globally injective and it is a piecewise-defined function composed of subfunctions $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$. Each function $\mathcal{F}_i$ maps a subset of X to Y. For the overall function $\mathcal{F}$ to qualify as a hesitant fuzzy function under Definition 3.1.1, each individual subfunction $\mathcal{F}_i$ must be injective on its respective domain. If any $\mathcal{F}_i$ fails to be injective, then the uniqueness of the image hesitant fuzzy set (condition 2 of Definition 3.1.1) may be violated, without a specific condition. The following example illustrates the necessity of the injectivity of $\mathcal{F}$ or its subfunctions $\mathcal{F}_i$.

Example 3.1.4. Let $A = \{\langle -1, \{0.3, 0.4, 0.2\}\rangle, \langle 1, \{0.1, 0.4, 0.5\}\rangle, \langle 2, \{0.5, 0.3, 0.7\}\rangle\}$ be an HFS on X, and let $\mathcal{F}(x) = x^2, and\ f_x(h(x)) = \{a^2 | a \in h(x)\}$ where the corresponding hesitant fuzzy function from X to Y is:

$$\begin{aligned}\mathbf{F}(A) &= \{\langle \mathcal{F}(x), f_x(h_A(x))\rangle | x \in X\} \\ &= \{\langle \mathcal{F}(-1), f_{-1}(\{0.3, 0.4, 0.2\})\rangle, \langle \mathcal{F}(1), f_1(\{0.1, 0.4, 0.5\})\rangle, \langle \mathcal{F}(2), f_2(\{0.5, 0.3, 0.7\})\rangle\} \\ &= \{\langle 1, f_{-1}(\{0.3, 0.4, 0.2\})\rangle, \langle 1, f_1(\{0.1, 0.4, 0.5\})\rangle, \langle 4, \{0.025, 0.09, 0.49\}\rangle\}\end{aligned}$$

which is not a hesitant fuzzy set on Y as $1 \in Y$ has two membership sets.

Lemma 3.1.2. Let $\mathbf{F} = \{\langle \mathcal{F}, f_x \rangle : x \in X\}$ be an HFF from X to Y. Suppose $\mathcal{F}(x_1) = \mathcal{F}(x_2) = y$ for some $x_1 \neq x_2 \in X$. Then, for any hesitant fuzzy set A on X, if $\mathbf{F}(A) = B$, it follows that

$$f_{x_1}(h_A(x_1)) = f_{x_2}(h_A(x_2)) = h_B(y).$$

Proof. By Remarks 3.1.1, $\mathbf{F}(A) = \{\langle \mathcal{F}(x), f_x(h_A(x))\rangle | \ x \in X\} = B = \{\langle y, h_B(y)\rangle | y \in Y\}$. Since $\mathcal{F}(x_1) = \mathcal{F}(x_2) = y$, both $f_{x_1}(h_A(x_1))$ and $f_{x_2}(h_A(x_2))$ must map to the same hesitant fuzzy element $h_B(y)$ to ensure B is well defined (i.e., there is only one image value assigned to y). Hence $f_{x_1}(h_A(x_1)) = f_{x_2}(h_A(x_2)) = h_B(y)$. □

Let $\mathbf{F}(A) = B$ and suppose there exists $x_1, x_2 \in X$ with $x_1 \neq x_2$, such that $\mathcal{F}(x_1) = \mathcal{F}(x_2) = y$. By Lemma 3.1.2, it follows that $f_{x_1}(h_A(x_1)) = f_{x_2}(h_A(x_2)) = h_B(y)$, which implies that the two distinct pairs $\langle x_1, h_A(x_1)\rangle \in A$ and $\langle x_2, h_A(x_2)\rangle \in A$ are mapped to the same output pair

$$\langle \mathcal{F}(x_1), f_{x_1}(h_A(x_1))\rangle = \langle \mathcal{F}(x_2), f_{x_1}(h_A(x_1))\rangle = \langle y, h_B(y)\rangle \in B.$$

This leads to loss of distinguishability, where the individual identities and hesitant characteristics of x_1 and x_2 are no longer preserved in the output. However, we are not interested in loss of distinguishability because hesitation is meant to reflect the unique uncertainty of each element, and losing that distinction makes the hesitant fuzzy function meaningless. So, we conclude a strong statement in the following remark:

Remark 3.1.2. To avoid loss of distinguishability and ensure that the image under **F** remains a valid hesitant fuzzy set, $\mathcal{F}$ must be injective or piecewise injective.

From this point onward, we shall assume that all HFFs under consideration are injective or piecewise injective to preserve the semantics of hesitation and ensure the validity of image sets.

The nondecreasing property is crucial in fuzzy set theory and in our work, as it ensures the preservation of ordering when transforming hesitant fuzzy elements. This property guarantees that if one hesitant fuzzy element has a lower mean value than another, their transformed values maintain the same order. Such consistency is essential in decision-making, ranking systems, and mathematical modelling, preventing contradictions and ensuring logical reasoning. In our framework, the nondecreasing property helps maintain stability and coherence in hesitant fuzzy functions, making them more reliable for practical applications. In addition, conditions 3 and 4 in Definition 3.1.1 ensure that f_x is nondecreasing on $P(I)$, as formalized in the following definition:

Definition 3.1.2. Let f_x be a co-membership function associated with an HFF **F**. We say that f_x is nondecreasing on $P(I)$ if, for any two HFEs $h_A(x)$ and $h_B(x)$ in $P(I)$,

$$mean(h_A(x)) \leq mean(h_B(x)), \quad \text{then } mean\left(f_x\big(h_A(x)\big)\right) \leq mean\left(f_x\big(h_B(x)\big)\right).$$

Moreover, by condition 4 of Definition 3.1.1, we have

$$f_x\big(h(x)\big) = \{0\} \text{ whenever } h(x) = \{0\}, \text{ and } f_x\big(h(x)\big) = \{1\} \text{ whenever } h(x) = \{1\}.$$

The converse of this implication is also true, ensuring boundary preservation.

Lemma 3.1.3. The co-membership function f_x is always nondecreasing.

Proof. This follows directly from conditions 3 and 4 of Definition 3.1.1.

To illustrate the nondecreasing property of f_x, let us consider the following example:

Example 3.1.5: Let us consider the HFF given in Example 3.1.1, $x = 1 \in X$ and $y = 2 \in Y$. Therefore

$$mean\big(h_A(1)\big) = mean\{0.4, 0.5, 0.7\} = 0.5\overline{3}$$

$$mean\big(h_C(1)\big) = mean\{0.1, 0.3\} = 0.2.$$

So,

$$mean\big(h_C(1)\big) \leq mean\big(h_A(1)\big).$$

Consequently,

$$mean\left(f_1\big(h_A(1)\big)\right) = mean\big(h_B(2)\big) = mean\{0.16, 0.25, 0.49\} = 0.3$$

and

$$mean\left(f_1\big(h_C(1)\big)\right) = mean\big(h_D(2)\big) = mean\{0.01, 0.09\} = 0.05.$$

Therefore,

$$mean\left(f_1\big(h_C(1)\big)\right) \leq mean\left(f_1\big(h_A(1)\big)\right)$$

and this example verified our statement in Lemma 3.1.3. Therefore, the function f_1 is nondecreasing. Similarly, all other values of $x \in X$ and $y \in Y$ satisfy the condition. Therefore, the function f_x is nondecreasing over its entire domain. This lemma essentially ensures consistency in the output of an HFF even when two distinct inputs map to the same output under **F**, aligning with the requirement of a well-defined hesitant fuzzy image.

In this section, we have formally introduced hesitant fuzzy functions (HFFs) as hesitant fuzzy relations that satisfy four key conditions. These conditions ensure well-defined behavior, preserve order and boundaries, and support consistent mappings between hesitant fuzzy sets. The notion of co-membership functions further refines the structure of HFFs, enabling a functional representation that captures both the domain and the range uncertainty.

3.2. *Basic properties of Hesitant Fuzzy Functions*

In classical function theory, properties such as equality, injectivity, and surjectivity serve as foundational concepts for studying mappings between sets. In the hesitant fuzzy context, these properties must be carefully adapted to account for multiple membership degrees arising from hesitation. This section explores the basic characteristics of hesitant fuzzy functions (HFFs), focusing on equality, image and preimage mappings, and injectivity and surjectivity.

Definition 3.2.1. Let $\mathbf{F}: P(I)^X \to P(I)^Y$ and $\mathbf{G}: P(I)^X \to P(I)^Y$ be two hesitant fuzzy functions from X to Y where $\mathbf{F} = \{\langle \mathcal{F}(x), f_x(h(x))\rangle : x \in X\}$ and $\mathbf{G} = \{\langle \mathcal{G}(x), g_x(h(x))\rangle : x \in X\}$. Then **F** is said to be equal to **G**, denoted by $\mathbf{F} = \mathbf{G}$, if and only if $\mathcal{F}(x) = \mathcal{G}(x)$ and $f_x(h(x)) = g_x(h(x))$, for all $x \in X$ and $h(x) \in P(I)$.

Lemma 3.2.1. Let $\mathbf{F} = \{\langle \mathcal{F}(x), f_x(h(x))\rangle | x \in X\}$ and $\mathbf{G} = \{\langle \mathcal{G}(x), g_x(h(x))\rangle | x \in X\}$ be two hesitant fuzzy functions from X to Y. Then $\mathbf{F} = \mathbf{G}$ if and only if $\mathbf{F}(A) = \mathbf{G}(A)$, for all HFS A on X, where $\mathbf{F}(A)$ and $\mathbf{G}(A)$ denote the images of A under **F** and **G**, respectively.

Proof. Firstly, let $\mathbf{F} = \mathbf{G}$ where $\mathbf{F} = \{\langle \mathcal{F}(x), f_x(h(x))\rangle | x \in X\}$ and $\mathbf{G} = \{\langle \mathcal{G}(x), g_x(h(x))\rangle | x \in X\}$. So $\mathcal{F}(x) = \mathcal{G}(x)$ and $f_x(h(x)) = g_x(h(x))$, for all $x \in X$ and $h(x) \in P(I)$. Let A be an HFS on X. From Remark 3.1.1,

$$\begin{aligned}\mathbf{F}(A) &= \{\langle \mathcal{F}(x), f_x(h_A(x))\rangle |\, x \in X\} \\ &= \{\langle \mathcal{G}(x), g_x(h_A(x))\rangle |\, x \in X\} = \mathbf{G}(A).\end{aligned}$$

Therefore $\mathbf{F}(A) = \mathbf{G}(A)$.

Secondly, let $\mathbf{F}(\mathrm{A}) = \mathbf{G}(A)$, where $A = \{\langle x, h(x)\rangle | x \in X\}$ be an HFS on X. Thus $\{\langle \mathcal{F}(x), f_x(h(x))\rangle | x \in X\} = \{\langle \mathcal{G}(x), g_x(h(x))\rangle | x \in X\}$. Hence $\mathcal{F}(x) = \mathcal{G}(x)$ and $f_x(h(x)) = g_x(h(x))$. Therefore $\mathbf{F} = \mathbf{G}$. □

Example 3.2.1. Let $A = \{\langle 1, \{0.5, 0.4, 0.7\}\rangle, \langle 2, \{0.2, 0.8, 0.3\}\rangle\}$ be an HFS on X, and let $\boldsymbol{F} = \{\langle \mathcal{F}, f_x\rangle | x \in X\}$ and $\mathbf{G} = \{\langle \mathcal{G}, g_x\rangle | x \in X\}$ be two hesitant fuzzy functions from X to Y, so $\mathcal{F}(x) = |x|$ and

$$f_x(h(x)) = \{\max(h(x))\} \quad \text{and} \quad \mathcal{G}(x) = \begin{cases} \mathcal{G}(x) = -x & if \quad -\infty < x < 0 \\ \mathcal{G}(x) = x & if \quad 0 \le x < \infty \end{cases}$$

and $g_x(h(x)) = \{\max(h(x))\}$, we have

$$\mathbf{F}(A) = \{\langle \mathcal{F}(x), f_x(h_A(x))\rangle | x \in X\}$$
$$= \{\langle \mathcal{F}(1), f_1\{0.5, 0.4, 0.7\}\rangle, \langle \mathcal{F}(2), f_2\{0.2, 0.8, 0.3\}\rangle\} = \{\langle 1, \{0.7\}\rangle, \langle 2, \{0.8\}\rangle\}.$$
$$\mathbf{G}(A) = \{\langle \mathcal{G}(x), g_x(h_A(x))\rangle | x \in X\}$$
$$= \{\langle \mathcal{G}(1), g_1\{0.5, 0.4, 0.7\}\rangle, \langle \mathcal{G}(2), g_2\{0.2, 0.8, 0.3\}\rangle\} = \{\langle 1, \{0.7\}\rangle, \langle 2, \{0.8\}\rangle\}.$$

$\mathcal{F}(1) = \mathcal{G}(1)$ and $\mathcal{F}(2) = \mathcal{G}(2)$ as well as $f_1(\{0.5, 0.4, 0.7\}) = g_1(\{0.5, 0.4, 0.7\})$ and $f_2(\{0.2, 0.8, 0.3\}) = g_2(\{0.2, 0.8, 0.3\})$. So $\mathcal{F}(x) = \mathcal{G}(x)$ and $f_x(h_A(x)) = g_x(h_A(x))$, for all $x \in X$. So, $\mathbf{F}(A) = \mathbf{G}(A)$. Therefore, $\mathbf{F} = \boldsymbol{G}$.

Definition 3.2.2. Let $\mathbf{F} = \{\langle \mathcal{F}, f_x\rangle | x \in X\}: P(I)^X \to P(I)^Y$ be a hesitant fuzzy function from X to Y. For an HFS B on Y, the preimage of HFS B under $\mathbf{F}$, denoted by $\mathbf{F}^{-1}(B)$, is the collection of all hesitant fuzzy sets A on X such that

$$\mathbf{F}(A) = B.$$

Example 3.2.2. From Example 3.1.1, $\mathbf{F}^{-1}(B) = \{A, E\}$.

Lemma 3.2.2 Let $\mathbf{F} = \{\langle \mathcal{F}, f_x\rangle | x \in X\}: P(I)^X \to P(I)^Y$ be a hesitant fuzzy function from X to Y. For every HFS B on Y, an HFS A on X is in $\mathbf{F}^{-1}(B)$, if and only if, for every $x \in X$, $x \in \mathcal{F}^{-1}(y)$ and $h_A(x) \in f_x^{-1}(h_B(y))$.

Proof. Firstly, let $A \in \mathbf{F}^{-1}(B)$, where A and B are HFSs on X and Y respectively. So $\mathbf{F}(A) = B$.

From Remark 3.1.1, if $\mathcal{F}(x) = y$ then $f_x(h_A(x)) = h_B(y)$. Therefore, if $x \in \mathcal{F}^{-1}(y)$ then $h_A(x) \in f_x^{-1}(h_B(y))$ (from the definition of the preimage of an ordinary function).

Secondly, let $x \in \mathcal{F}^{-1}(y)$ and $h_A(x) \in f_x^{-1}(h_B(y))$. So $\mathcal{F}(x) = y$ and $f_x(h_A(x)) = h_B(y)$. So $\mathbf{F}(A) = B$. Therefore, $A \in \mathbf{F}^{-1}(B)$. □

From Example 3.2.2, it is obvious that $A \in \mathbf{F}^{-1}(B)$. From Example 3.1.2, $2 \in \mathcal{F}^{-1}(4)$ and $h_A(2) = \{0.4, 0.7, 0.6\} \in f_2^{-1}(h_{F(A)}(4) = f_2^{-1}(h_B)(4) = f_2^{-1}(\{0.16, 0.25, 0.49\}) = \{0.4,0.7, 0.6\}$. This verifies the Lemma 3.2.2.

We conclude that a hesitant fuzzy function $\mathbf{F}$ acts as a mapping from an HFS A on X to an HFS B on Y. Under this framework, the image of an HFS under HFF is itself an HFS. Consequently, the operations of inclusion, union, and intersection of two images of two HFSs under such hesitant fuzzy functions correspond to the respective operations on hesitant fuzzy sets. This correspondence ensures that the algebraic structure of hesitant fuzzy sets is preserved under hesitant fuzzy functions, allowing us to extend classical fuzzy set operations to the hesitant fuzzy setting in a functionally consistent manner.

Lemma 3.2.3. Let $\mathbf{F} = \{\langle \mathcal{F}(x), f_x(h(x))\rangle | x \in X\}$ be a hesitant fuzzy function from X to Y. For every two HFS A and B on X, $\mathbf{F}(A)$ is a subset of $\mathbf{F}(B)$ (denoted as $\mathbf{F}(A) \subseteq \mathbf{F}(B)$), if and only if

$$mean\left(f_x(h_A(x))\right) \leq mean\left(f_x(h_B(x))\right), \text{ for all } x \in X.$$

Also, $\mathbf{F}(A)$ is a proper subset of $\mathbf{F}(B)$, denoted as $\mathbf{F}(A) \subset \mathbf{F}(B)$, if and only if

$$mean\left(f_x(h_A(x))\right) < mean\left(f_x(h_B(x))\right), \text{ for all } x \in X.$$

Proof. The proof follows directly from the definition of a subset of two HFSs in Definition 2.7. □

Theorem 3.2.1. Let $\mathbf{F} = \{\langle \mathcal{F}, f_x\rangle | x \in X\}: P(I)^X \to P(I)^Y$ be a hesitant fuzzy function from X to Y. For all hesitant fuzzy sets A, B of X and for all hesitant fuzzy sets C, D of Y, we have:

1) $\mathbf{F}(\emptyset^*) = \emptyset^*$,
2) $\mathbf{F}(O^*) = O^*$,
3) If $A \subset B$ then $\mathbf{F}(A) \subset \mathbf{F}(B)$,
4) $\mathbf{F}(A \cup B) = \mathbf{F}(A) \cup \mathbf{F}(B)$,
5) $\mathbf{F}(A \cap B) = \mathbf{F}(A) \cap \mathbf{F}(B)$,
6) If $C \subset D$ and f_x^{-1} is a nondecreasing inverse function of f_x, then $\mathbf{F}^{-1}(C) \subset \mathbf{F}^{-1}(D)$. Without the nondecreasing property of f_x^{-1} $\mathbf{F}^{-1}(C) \subset \mathbf{F}^{-1}(D)$ is not necessarily true.
7) $A \in \mathbf{F}^{-1}(\mathbf{F}(A))$,
8) $\mathbf{F}(\mathbf{F}^{-1}(C)) = \{C\}$,
9) Let $C, D \in P(I)^Y$. Then $C = D$ if and only if $\mathbf{F}^{-1}(C) = \mathbf{F}^{-1}(D)$.

Proof.

1) Since $\phi^* = \{\langle x, \phi\rangle | \, x \in X\}$ is a nonsense HFS on X, then $\mathbf{F}(\phi^*) = \{\langle \mathcal{F}(x), f_x(\phi)\rangle | \, x \in X\} = \{\langle \mathcal{F}(x), \phi\rangle | \, x \in X\} = \phi^*$, where ϕ^* is a nonsense HFS on Y.

2) Since $O^* = \{\langle x, \{0\}\rangle | \, x \in X\}$ is an empty HFS on X, then $\mathbf{F}(O^*) = \{\langle \mathcal{F}(x), f_x(\{0\})\rangle | x \in X\} = \{\langle \mathcal{F}(x), \{0\}\rangle | x \in X\} = O^*$, where O^* is an empty HFS on Y.

3) Let A and B be two HFSs on X, such that $A \subset B$. So $mean\left(h_A(x)\right) < mean\left(h_B(x)\right)$. Now let $\mathbf{F}(A) = C$ and $\mathbf{F}(B) = D$ are HFSs on Y. So $A \dot{\times} C \in \mathbf{F}$ and $B \dot{\times} D \in \mathbf{F}$. So, from the nondecreasing property for f_x, $mean\left(h_C(x)\right) < mean\left(h_D(x)\right)$. So $C \subset D$. Therefore, $\mathbf{F}(A) \subset \mathbf{F}(B)$.

4)
$$\begin{aligned}\mathbf{F}(A \cup B) &= \mathbf{F}(\{\langle x, h_{A\cup B}(x)\rangle | x \in X\}) \\ &= \{\langle \mathcal{F}(x), f_x(h_{A\cup B}(x))\rangle | x \in X\} \\ &= \{\langle \mathcal{F}(x),\ f_x(\{\delta \in h_A(x) \cup h_B(x) | \delta \geq \max\{h_A^-(x), h_B^-(x)\}\})\rangle | x \in X\}.\end{aligned}$$

Now let $\mathbf{F}(A) = \{\langle \mathcal{F}(x),\ f_x(h_A(x))\rangle | x \in X\} = \{\langle y, h_C(y)\rangle | y \in Y\} = C$, and let $\mathbf{F}(B) = \{\langle \mathcal{F}(x),\ f_x(h_B(x))\rangle | x \in X\} = \{\langle y, h_D(y)\rangle | y \in Y\} = D$. So $\mathcal{F}(x) = y$, $f_x(h_A(x)) = h_C(y)$ and $f_x(h_B(x)) = h_D(y)$.Therefore,

$$\begin{aligned}\mathbf{F}(A) \cup \mathbf{F}(B) &= \{\langle y, h_C(y)\rangle | y \in Y\} \cup \{\langle y, h_D(y)\rangle | y \in Y\} \\ &= \{\langle y, h_{C\cup D}(y)\rangle | y \in Y\}\end{aligned}$$

where $h_{C\cup D}(y) = \{\delta \in h_C(y) \cup h_D(y) | \delta \geq \max\{h_C^-(y), h_D^-(y)\} | y \in Y\}$. Using the fact that f_x is nondecreasing, and therefore $f_x(h_A(x)) = h_C(x)$, implies $f_x(h_A^-(x)) = h_C^-(x)$, and therefore we now have the following

$$\begin{aligned}h_{C\cup D}(y) &= \{\delta \in f_x(h_A(x)) \cup f_x(h_B(x)) \big| \delta \geq \max\{f_x(h_A^-(x)), f_x(h_B^-(x))\} \big| y \in Y\} \\ &= \{\delta \in (h_A(x)) \cup f_x(h_B(x)) | \delta \geq \max\{h_A^-(y), h_B^-(y)\} | y \in Y\}.\end{aligned}$$

Since $\mathcal{F}(x) = y$ and $f_x\{\delta \in (h_A(x)) \cup f_x(h_B(x)) | \delta \geq \max\{h_A^-(y), h_B^-(y)\} | y \in Y\} = h_{C\cup D}(y)$, then $\mathbf{F}(A \cup B) = \mathbf{F}(A) \cup \mathbf{F}(B)$.

5) The proof is analogous to that of (4).

6) Let $C \subset D$ be HFSs on Y, and $\mathbf{F}$ be an HFF from X to Y with nondecreasing f_x^{-1}. Since $C \subset D$, so $mean(h_C(y)) < mean(h_D(y))$. Now let $A, B \in P(I)^X$ such that $A \in \mathbf{F}^{-1}(C)$ and $B \in \mathbf{F}^{-1}(D)$. So $\mathbf{F}(A) = C$ and $\mathbf{F}(B) = D$. So $A \dot{\times} C, B \dot{\times} D \in \mathbf{F}$. Then, from the nondecreasing property of the inverse of f_x, $mean\left(h_C(x)\right) < mean\left(h_D(x)\right)$ implies $mean\left(f_x^{-1}(h_C(x))\right) < mean\left(f_x^{-1}(h_D(x))\right)$. Since f_x and f_x^{-1} are bijective, then $f_x^{-1}(h_C(x)) = h_A(x)$ and $f_x^{-1}(h_D(x)) = h_B(x)$. So $mean(h_A(x)) < mean(h_B(x))$. So $A \subset B$. Therefore, $\mathbf{F}^{-1}(C) \subset \mathbf{F}^{-1}(D)$. For the second part, let $C \subset D \in P(I)^Y$ and the inverse of f_x does not exist. On the contrary assume $\mathbf{F}^{-1}(C) \subset \mathbf{F}^{-1}(D)$. Let $A, B \in P(I)^X$ such that $A \in \mathbf{F}^{-1}(C)$ and $B \in \mathbf{F}^{-1}(D)$. So $\mathbf{F}(A) = C$ and $\mathbf{F}(B) = D$. Since $\mathbf{F}^{-1}(C) \subset \mathbf{F}^{-1}(D)$ (the normal subset because we consider the set with a collection of some HFSs) then $A \in \mathbf{F}^{-1}(D)$. So $\mathbf{F}(A) = D$. This contradicts the fact that $\mathbf{F}(A) = C$ and the image of HFF is always unique.

7) Let A be an HFS on X and let $\mathbf{F}: P(I)^X \to P(I)^Y$ be a hesitant fuzzy function from X to Y. Assume that $\mathbf{F}(A) = C$, where C is an HFS on Y. By the definition of the preimage of HFS C under $\mathbf{F}$, we have

$$\mathbf{F}^{-1}(C) = \{B \in \mathrm{P}(I)^{\mathrm{X}} : \mathbf{F}(B) = C\}$$

Since $\mathbf{F}(A) = C$, it follows directly that $A \in \{B \in \mathrm{P}(I)^{\mathrm{X}} : \mathbf{F}(B) = C\}$, which implies that $A \in \mathbf{F}^{-1}(C)$. Therefore, $A \in \mathbf{F}^{-1}(\mathbf{F}(A))$. However, the set $\mathbf{F}^{-1}(\mathbf{F}(A))$ is not necessarily a singleton containing only A. The following Counterexample illustrates that $\mathbf{F}^{-1}(\mathbf{F}(A))$ may contain more than one HFS, even when $\mathbf{F}(A) = \mathbf{F}(B)$ but $A \neq B$.

Counterexample

Let $X = \{1, 2\}$ and $Y = \{2, 4\}$ be universal sets, and let $A = \{\langle 1, \{0.4, 0.6\}\rangle, \langle 2, \{0.1, 0.7\}\rangle\}$ and $B = \{\langle 1, \{0.3, 0.6\}\rangle, \langle 2, \{0.7\}\rangle\}$ be HFSs on X. Let $\mathbf{F} = \{\langle \mathcal{F}, f_x\rangle |\ \mathrm{x} \in \mathrm{X}\} : P(I)^X \to P(I)^Y$ be a hesitant fuzzy function from X to Y, defined as follows:

Let $\mathcal{F}(x) = 2x$ and $f_x(U) = \begin{cases} \{\max(h(x))\} & if \quad U = h(x) \\ \phi & if \quad U \neq h(x) \end{cases}$, then we have

$$\mathbf{F}(A) = \mathbf{F}(\{\langle 1, \{0.4, 0.6\}\rangle, \langle 2, \{0.1, 0.7\}\rangle\}) = \{\langle 2, \{0.6\}\rangle, \langle 4, \{0.7\}\rangle\} = C, \text{ and}$$

$$\mathbf{F}(B) = \mathbf{F}(\{\langle 1, \{0.3, 0.6\}\rangle, \langle 2, \{0.7\}\rangle\}) = \{\langle 2, \{0.6\}\rangle, \langle 4, \{0.7\}\rangle\} = C,$$

where $C \in P(I)^Y$ is an HFS on Y.

Although $A \neq B$, we have $\mathbf{F}(A) = \mathbf{F}(B) = C$. Therefore, both $A, B \in \mathbf{F}^{-1}(C) = \mathbf{F}^{-1}(\mathbf{F}(A))$, showing that:

$$\mathbf{F}^{-1}(\mathbf{F}(A)) = \{A, B\}.$$

This confirms that $A \in \mathbf{F}^{-1}(\mathbf{F}(A))$, but the preimage set contains more than just A.

8) Since $\mathbf{F}^{-1}(C) = \{B \in P(I)^X | \mathbf{F}(B) = C\}$, then

$$\begin{aligned} \mathbf{F}(\mathbf{F}^{-1}(C)) &= \mathbf{F}(\{B \in P(I)^X | \mathbf{F}(B) = C\}) \\ &= \{\mathbf{F}(B) : B \in P(I)^Y, \mathbf{F}(B) = C\} \\ &= \{C\}. \end{aligned}$$

9) ($\Longrightarrow$) Suppose that $C = D$, then $\mathbf{F}^{-1}(C) = \{A \in P(I)^X |\ \mathbf{F}(A) = C\}$. Since $C = D$, then $\mathbf{F}^{-1}(C) = \{A \in P(I)^X |\ \mathbf{F}(A) = D\} = \mathbf{F}^{-1}(D)$. Therefore, $\mathbf{F}^{-1}(C) = \mathbf{F}^{-1}(D)$.
($\Longleftarrow$) Suppose that $\mathbf{F}^{-1}(C) = \mathbf{F}^{-1}(D)$. Since $\mathbf{F}^{-1}(C) = \{A \in P(I)^X |\ \mathbf{F}(A) = C\}$, then for any $B \in \mathbf{F}^{-1}(C)$, $\mathbf{F}(B) = C$, but since $\mathbf{F}^{-1}(C) = \mathbf{F}^{-1}(D)$, $B \in \mathbf{F}^{-1}(D)$. Hence, $\mathbf{F}(B) = D$. Thus, $C = \mathbf{F}(B) = D$. So, $C = D$. □

This section formalizes key structural properties of hesitant fuzzy functions, including function equality, image and preimage behavior, and injectivity under additional assumptions. Taken together, these properties show that HFFs provide a well-structured generalization of classical functions and support consistent analysis in uncertain environments with hesitant membership information.

3.3. *Composition and invertibility of Hesitant Fuzzy Functions*

This section investigates the compositional and invertibility properties of hesitant fuzzy functions. We establish that composition of HFFs is associative, characterize injectivity and surjectivity in terms of their component functions, and show that every bijective HFF admits a unique inverse.

Definition 3.3.1. Let $\mathbf{F} = \{\langle \mathcal{F}, f_x\rangle | x \in X\}: P(I)^X \to P(I)^Y$ and $\mathbf{G} = \{\langle \mathcal{G}, g_y\rangle | y \in Y\}: P(I)^Y \to P(I)^Z$ be two hesitant fuzzy functions from X to Y and Y to Z, respectively, then the composition of **F** and **G**, denoted by $\mathbf{G} \circ \mathbf{F}$, is the hesitant fuzzy function from X to Z, defined for all $A \in P(I)^X$ as:

$$(\mathbf{G} \circ \mathbf{F})(A) = \mathbf{G}(\mathbf{F}(A)) = \{\langle (\mathcal{G} \circ \mathcal{F})(x), (g_{\mathcal{F}(x)} \circ f_x)(h_A(x))\rangle | x \in X\}$$
$$= \left\{\langle \mathcal{G}(\mathcal{F}(x)), g_{\mathcal{F}(x)}\left(f_x(h_A(x))\right)\rangle \, | x \in X\right\}.$$

Example 3.3.1. Let X, Y and Z be universal sets, and let $A = \{\langle 1, \{0.5, 0.6, 0.8\}\rangle, \langle 2, \{0.2, 0.4, 0.9\}\rangle, \langle 3, \{0.4, 0.6, 0.8\}\rangle\}$ be an HFS on X, and let $\mathbf{F} = \{\langle \mathcal{F}(x), f_x(h_A(x))\rangle | x \in X\}: P(I)^X \to P(I)^Y$ be a hesitant fuzzy function from X to Y, where $\mathcal{F}(x) = 2x$ and $f_x(h_A(x)) = \left\{\frac{a}{2}: a \in h(x)\right\}$. Let $\mathbf{G} = \{\langle \mathcal{G}(y), f_y(h_B(y))\rangle | x \in X\}: P(I)^Y \to P(I)^Z$ be a hesitant fuzzy function from Y to Z, where $\mathcal{G}(y) = 3y$ and $g_y(h_B(y)) = \{a^2: a \in h(y)\}$. Then the composition of **F** and **G** (denoted by $\mathbf{G} \circ \mathbf{F}$**)** is the hesitant fuzzy function from X to Z, as follows:

$$(\mathbf{G} \circ \mathbf{F})(A) = \mathbf{G}(\mathbf{F}(A)) = \{\langle (\mathcal{G} \circ \mathcal{F})(x), (g_{\mathcal{F}(x)} \circ f_x)(h(x))\rangle | x \in X\}$$
$$= \left\{\langle \mathcal{G}(\mathcal{F}(x)), g_{\mathcal{F}(x)}\left(f_x(h(x))\right)\rangle \, | x \in X\right\}$$
$$= \{\langle \mathcal{G}(y), g_y(h(y))\rangle | y \in Y\} = C,$$

which is an HFS on Z.

$$\mathbf{F}(A) = \{\langle \mathcal{F}(1), f_1(\{0.5, 0.6, 0.8\})\rangle, \langle \mathcal{F}(2), f_2(\{0.2, 0.4, 0.9\})\rangle, \langle \mathcal{F}(3), f_3(\{0.4, 0.6, 0.8\})\rangle\}$$
$$= \{\langle 2, \{0.25, 0.3, 0.4\}\rangle, \langle 4, \{0.1, 0.2, 0.45\}\rangle, \langle 6, \{0.2, 0.3, 0.4\}\rangle\} = B,$$

which is an HFS on Y. Therefore, $A \dot{\times} B \in \mathbf{F}$.

$$\mathbf{G}(\mathbf{F}(A)) = \mathbf{G}(B)$$
$$= \{\langle \mathcal{G}(2), g_2(\{0.25, 0.3, 0.4\})\rangle, \langle \mathcal{G}(4), g_4(\{0.1, 0.2, 0.45\})\rangle, \langle \mathcal{G}(6), g_6(\{0.2, 0.3, 0.4\})\rangle\}$$
$$= \{\langle 6, \{0.0625, 0.09, 0.16\}\rangle, \langle 12, \{0.01, 0.04, 0.2025\}\rangle, \langle 18, \{0.04, 0.09, 0.16\}\rangle\}$$
$$= C,$$

which is an HFS on Z. $\mathbf{G}(B) = C$, therefore, $B \dot{\times} C \in \mathbf{G}$. $(\mathbf{G} \circ \mathbf{F})(A) = \mathbf{G}(\mathbf{F}(A)) = \mathbf{G}(B) = C$, therefore $A \dot{\times} C \in \mathbf{G} \circ \mathbf{F}$.

Theorem 3.3.1. Let $\mathbf{F}: P(I)^X \to P(I)^Y$, $\mathbf{G}: P(I)^Y \to P(I)^Z$ and $\mathbf{K}: P(I)^Z \to P(I)^W$ be hesitant fuzzy functions. Then, the composition of hesitant fuzzy functions is associative, that is:

$$(\mathbf{K} \circ \mathbf{G}) \circ \mathbf{F} = \mathbf{K} \circ (\mathbf{G} \circ \mathbf{F}).$$

Proof. Let $\mathbf{F} = \{\langle \mathcal{F}, f_x\rangle | x \in X\}: P(I)^X \to P(I)^Y$, $\mathbf{G} = \{\langle \mathcal{G}, g_y\rangle | y \in Y\}: P(I)^Y \to P(I)^Z$ and $\mathbf{K} = \{\langle \mathcal{K}, k_z\rangle | z \in Z\}: P(I)^Z \to P(I)^W$ be hesitant fuzzy functions, and $A \in P(I)^X$ be an HFS.

Firstly, consider the left-hand side:

$$
\begin{aligned}
\big((\mathbf{K}\circ\mathbf{G})\circ\mathbf{F}\big)(A) &= (\mathbf{K}\circ\mathbf{G})\big(\mathbf{F}(A)\big)\\
&= (\mathbf{K}\circ\mathbf{G})\big\{\langle(\mathcal{F}(x)), f_x\big(h(x)\big)\rangle|x\in X\big\}\\
&= \mathbf{K}\big(\mathbf{G}\big\{\langle(\mathcal{F}(x)), f_x\big(h(x)\big)\rangle\big\}\big)\\
&= \mathbf{K}\left(\left\{\langle\mathcal{G}\big(\mathcal{F}(x)\big), g_{\mathcal{F}(x)}\left(f_x\big(h(x)\big)\right)\rangle\,|x\in X\right\}\right)\\
&= \left\{\langle\mathcal{K}\left(\mathcal{G}\big(\mathcal{F}(x)\big)\right), k_{\mathcal{G}(\mathcal{F}(x))}\left(g_{\mathcal{F}(x)}\left(f_x\big(h(x)\big)\right)\right)\rangle\right\}
\end{aligned}
$$

Now, consider the right-hand side:

$$
\begin{aligned}
\big(\mathbf{K}\circ(\mathbf{G}\circ\mathbf{F})\big)(A) &= \mathbf{K}\big((\mathbf{G}\circ\mathbf{F})(A)\big)\\
&= \mathbf{K}\left(\left\{\langle\mathcal{G}\big(\mathcal{F}(x)\big), g_{\mathcal{F}(x)}\left(f_x\big(h(x)\big)\right)\rangle\,|x\in X\right\}\right)\\
&= \left\{\langle\mathcal{K}\left(\mathcal{G}\big(\mathcal{F}(x)\big)\right), k_{\mathcal{G}(\mathcal{F}(x))}\left(g_{\mathcal{F}(x)}\left(f_x\big(h(x)\big)\right)\right)\rangle\,|x\in X\right\}\\
&= \big((\mathbf{K}\circ\mathbf{G})\circ\mathbf{F}\big)(A).
\end{aligned}
$$

Therefore, $(\mathbf{K}\circ\mathbf{G})\circ\mathbf{F} = \mathbf{K}\circ(\mathbf{G}\circ\mathbf{F})$, and the composition of HFFs is associative. □

Definition 3.3.2. A hesitant fuzzy function $\mathbf{F} = \{\langle\mathcal{F}, f_x\rangle|\, x\in X\}: P(I)^X \to P(I)^Y$ is said to be injective if and only if for all $A, B\in P(I)^X$,

$$\mathbf{F}(A) = \mathbf{F}(B) \quad \text{implies} \quad A = B.$$

Here, equality $A = B$ is understood in the strict sense, that is, $h_A(x) = h_B(x)$ for all $x\in X$, which is stronger than the mean-based equality of Definition 2.8, which is used exclusively for comparison and inclusion purposes.

Theorem 3.3.2. A hesitant fuzzy function $\mathbf{F}: P(I)^X \to P(I)^Y$ is injective if and only if both $\mathcal{F}$ and f_x are injective.

Proof. Firstly, let $\mathbf{F}: P(I)^X \to P(I)^Y$ be injective. From Remark 3.1.2 $\mathcal{F}$ is either injective or piecewise injective. Without loss of generality, we assume that $\mathcal{F}$ is injective. Now for any $x\in X$, let $f_x(U) = f_x(V)$ where $U = h_A(x)$ and $V = h_B(x)$ where

$$A = \{\langle x, h_A(x)\rangle|x\in X\} \quad \text{and} \quad B = \{\langle x, h_B(x)\rangle|x\in X\}.$$

So

$$\mathbf{F}(A) = \{\langle\mathcal{F}(x), f_x(h_A(x))\rangle|x\in X\} = \{\langle\mathcal{F}(x), f_x(h_B(x))\rangle|\, x\in X\} = \mathbf{F}(B).$$

Since $\mathbf{F}$ is injective, so $A = B$. That is, $h_A(x) = h_B(x)$ for all $x\in X$. So $U = V$ and therefore f_x is injective.

Secondly, let $\mathcal{F}$ and f_x for all $x\in X$ are injective. Suppose $\mathbf{F}(A) = \mathbf{F}(B)$ for some $A, B\in P(I)^X$, where $A = \{\langle x, h_A(x)\rangle|x\in X\}$ and $B = \{\langle x, h_B(x)\rangle|x\in X\}$. So $\mathcal{F}(x) = y$ and $f_x\big(h_A(x)\big) = f_x\big(h_B(x)\big)$. Since f_x is injective, then $h_A(x) = h_B(x)$ for all $x\in X$. hence $A = B$ and therefore $\mathbf{F}$ is injective. □

Definition 3.3.3. A hesitant fuzzy function $\mathbf{F} = \{\langle \mathcal{F}, f_x \rangle | \, x \in X\}: P(I)^X \to P(I)^Y$ is said to be surjective (onto) if and only if for all $B \in \mathrm{P(I)}^{\mathrm{Y}}$, there exists $A \in P(I)^X$ such that $\mathbf{F}(A) = B$.

Theorem 3.3.3. A hesitant fuzzy function $\mathbf{F}: P(I)^X \to P(I)^Y$ is surjective if and only if both $\mathcal{F}: X \to Y$ and $f_x: P(I)^X_{\text{finite}} \to P(I)^Y_{\text{finite}}$ are surjective, where $P(I)^A_{\text{finite}} \subset P(I)^A$ such that every set in $P(I)^A_{\text{finite}}$ is finite.

Proof. Firstly, let **F** be a surjective hesitant fuzzy function from X to Y, where $\mathbf{F} = \{\langle \mathcal{F}(x), f_x(h(x)) \rangle | \, x \in X\}$. Let $y_0 \in Y$ and $B \in P(I)^Y$ such that $h_B(y_0) \neq \emptyset$. Since **F** is surjective, then we let $A \in P(I)^X$ such that $\mathbf{F}(A) = B$. Therefore

$$\mathbf{F}(A) = \{\langle \mathcal{F}(x), f_x(h_A(x)) \rangle | x \in X\} = \{\langle y, h_B(y) \rangle | y \in Y\}.$$

So there exists $x_0 \in X$ such that $\mathcal{F}(x_0) = y_0$. So $\mathcal{F}$ is surjective. Now we let $x \in X$, $f_x: P(I)^X_{\text{finite}} \to P(I)^Y_{\text{finite}}$, $V \in P(I)$ and $B \in P(I)^Y_{\text{finite}}$, such that $V = h_B(y)$ for some $y \in Y$. Since **F** is surjective, then there exists $A \in P(I)^X$ such that $\mathbf{F}(A) = B$. Therefore

$$\mathbf{F}(A) = \{\langle \mathcal{F}(x), f_x(h_A(x)) \rangle | x \in X\} = \{\langle y, h_B(y) \rangle | y \in Y\}.$$

So $f_x(h_A(x)) = h_B(y) = V$. So f_x is surjective for the given domain and range.

Secondly, assume $\mathcal{F}$ and each f_x are surjective. Let B be arbitrary HFS on Y and for each $y \in Y$ we choose $x \in X$ with $\mathcal{F}(x) = y$ and U such that $f_x(U) = h_B(y)$ (possible by surjectivity of f_x). By define A by

$$h_A(x) = U.$$

Then $\mathbf{F}(A) = B$. Therefore **F** is surjective. □

Definition 3.3.4. A hesitant fuzzy function $\mathbf{F} = \{\langle \mathcal{F}, f_x \rangle | \, x \in X\}: P(I)^X \to P(I)^Y$ is said to be bijective if and only if it is both injective and surjective.

Lemma 3.3.1. A hesitant fuzzy function $\mathbf{F}: P(I)^X \to P(I)^Y$ is bijective if and only if both $\mathcal{F}$ and f_x are bijective functions.

Proof. The proof follows directly by combining Theorems 3.3.2 and 3.3.3. □

Definition 3.3.5. Let $\mathbf{F}: P(I)^X \to P(I)^Y$ be a hesitant fuzzy function from X to Y, and let $\mathbf{G}: P(I)^Y \to P(I)^X$ be a bijective hesitant fuzzy function from Y to X. If there exists such a hesitant fuzzy function **G** satisfying:

$$\mathbf{F} \circ \mathbf{G} = \mathbf{I} = \mathbf{G} \circ \mathbf{F},$$

where **I** is the hesitant fuzzy identity function on X, then **F** is called an invertible hesitant fuzzy function, and **G** is called the inverse of **F**, denoted as $\mathbf{F}^{-1} = \mathbf{G}$.

Note that the hesitant fuzzy identity function $\mathbf{I}: P(I)^X \to P(I)^X$ is defined by $\mathbf{I}(A) = A$ for any $A \in P(I)^X$. If **F** is invertible, then there exists exactly one hesitant fuzzy function **G** satisfying this property as shown in the following theorem;

Theorem 3.3.4. If $\mathbf{F}$ is an invertible hesitant fuzzy function, then the inverse $\mathbf{F}^{-1}$ is unique.

Proof. Let $\mathbf{F}: P(I)^X \to P(I)^Y$ be a hesitant fuzzy function from X to Y, and let $\mathbf{G}_1, \mathbf{G}_2: P(I)^Y \to P(I)^X$ be two hesitant fuzzy functions from Y to X such that both satisfy the inverse property of $\mathbf{F}$. That is,

$$\mathbf{F} \circ \mathbf{G_1} = \mathbf{I} \quad \text{and} \quad \mathbf{G_1} \circ \mathbf{F} = \mathbf{I}$$

$$\mathbf{F} \circ \mathbf{G}_2 = \mathbf{I} \quad \text{and} \quad \mathbf{G}_2 \circ \mathbf{F} = \mathbf{I}$$

where $\mathbf{I}$ is the identity hesitant fuzzy function, defined by:

$$\mathbf{I}(A) = A \text{ for all } A \in P(I)^X.$$

We want to show that $\mathbf{G}_1 = \mathbf{G}_2$.

Consider $\mathbf{G}_1 = \mathbf{G}_1 \circ \mathbf{I}$ (since $\mathbf{I}$ is the identity on its domain)

$= \mathbf{G}_1 \circ (\mathbf{F} \circ \mathbf{G}_2)$ (since $\mathbf{F} \circ \mathbf{G}_2 = \mathbf{I}$)

$= (\mathbf{G}_1 \circ \mathbf{F}) \circ \mathbf{G}_2$ (by associativity of composition)

$= \mathbf{I} \circ \mathbf{G}_2 = \mathbf{G}_2$ (since $\mathbf{G_1} \circ \mathbf{F} = \mathbf{I}$)

Hence, $\mathbf{G}_1 = \mathbf{G}_2$, so the inverse is unique.

Therefore, the inverse $\mathbf{F}^{-1}$ is unique. ☐

Theorem 3.3.5. Let $\mathbf{F} = \{\langle \mathcal{F}, f_x \rangle | x \in X\}: P(I)^X \to P(I)^Y$ be the hesitant fuzzy functions. Then $\mathbf{F}$ is invertible if and only if both $\mathcal{F}: X \to Y$ and $f_x: P(I)^X_{\text{finite}} \to P(I)^Y_{\text{finite}}$ are invertible functions. Moreover, the inverse of $\mathbf{F}$, denoted $\mathbf{F}^{-1}$, is given by:

$$\mathbf{F}^{-1} = \{\langle \mathcal{F}^{-1}, f_x^{-1} \rangle | x \in X\}$$

where $\mathcal{F}^{-1}: Y \to X$ and $f_y^{-1}: P(I)^Y_{\text{finite}} \to P(I)^X_{\text{finite}}$ are inverses for $\mathcal{F}$ and f_x respectively.

Proof. First, aassume that $\mathcal{F}$ and each f_x are invertible with inverse functions $\mathcal{F}^{-1}$ and f_x^{-1} respectively. Since $\mathcal{F}$ is bijective, there exists a bijection $\mathcal{F}^{-1}: Y \to X$ with

$$\mathcal{F}^{-1}(\mathcal{F}(x)) = x \quad \text{and} \qquad \mathcal{F}(\mathcal{F}^{-1}(y)) = y.$$

For each $y \in Y$, let $x = \mathcal{F}^{-1}(y)$ and define

$$f_y^{-1} \coloneqq f_x^{-1}.$$

Then

$$f_y^{-1}(h_B(y)) = f_x^{-1}(h_B(y)).$$

Now define a hesitant fuzzy function

$$\mathbf{F}^{-1} = \{\langle \mathcal{F}^{-1}, f_y^{-1} \rangle | y \in Y\}: P(I)^Y \to P(I)^X.$$

Let $B = \{\langle y, h_B(y) \rangle | y \in Y\} \in P(I)^Y$. We compute

$$\mathbf{F}^{-1}(B) = \{\langle \mathcal{F}^{-1}(y), f_y^{-1}(h_B(y))\rangle | y \in Y\}.$$

Applying **F**, we obtain

$$\begin{aligned}(\mathbf{F} \circ \mathbf{F}^{-1})(B) &= \mathbf{F}(\{\langle \mathcal{F}^{-1}(y), f_y^{-1}(h_B(y))\rangle | y \in Y\}) \\ &= \{\langle \mathcal{F}(\mathcal{F}^{-1}(y)), f_x(f_y^{-1}(h_B(y))\rangle | y \in Y\} \\ &= \left\{\langle \mathcal{F}(\mathcal{F}^{-1}(y)), f_x\left(f_x^{-1}(h_B(y))\right)\rangle \,|\, y \in Y\right\}, \qquad \text{(Since } f_y^{-1} \coloneqq f_x^{-1}\text{)} \\ &= \{\langle y, h_B(y)\rangle | y \in Y\} = B.\end{aligned}$$

A similar computation shows that $(\mathbf{F^{-1}} \circ \mathbf{F})(A) = A$ for all $A \in P(I)^X$, by symmetry of the argument above. Hence $\mathbf{F}^{-1}$ is the inverse of **F**, and therefore **F** is invertible.

Conversely, suppose $\mathbf{F}^{-1}$ exists. Then **F** is a bijective hesitant fuzzy function. By Lemma 4.2.7, this implies that the underlying map $\mathcal{F}: X \to Y$ and each $f_x: \mathcal{P}_{\text{fin}}(I) \to \mathcal{P}_{\text{fin}}(I)$ are bijective, and hence invertible. □

Theorem 3.3.6. For any two invertible hesitant fuzzy functions $\mathbf{F}: P(I)^X \to P(I)^Y$ and $\mathbf{G}: P(I)^Y \to P(I)^X$ we have:

$$(\mathbf{G} \circ \mathbf{F})^{-1} = \mathbf{F^{-1}} \circ \mathbf{G^{-1}}.$$

Proof. Let $\mathbf{F}: P(I)^X \to P(I)^Y$ and $\mathbf{G}: P(I)^Y \to P(I)^X$ be invertible HFFs with its inverse $\mathbf{F}^{-1}: P(I)^Y \to P(I)^X$ and $\mathbf{G}^{-1}: P(I)^X \to P(I)^Y$, respectively.

$$\begin{aligned}(\mathbf{G} \circ \mathbf{F}) \circ (\mathbf{F}^{-1} \circ \mathbf{G}^{-1}) &= \mathbf{G} \circ (\mathbf{F} \circ (\mathbf{F}^{-1} \circ \mathbf{G}^{-1})) \\ &= \mathbf{G} \circ ((\mathbf{F} \circ \mathbf{F}^{-1}) \circ \mathbf{G}^{-1}) \\ &= \mathbf{G} \circ I \circ \mathbf{G}^{-1} \\ &= \mathbf{G} \circ \mathbf{G}^{-1} \\ &= \mathbf{I}.\end{aligned}$$

The same goes to $(\mathbf{F}^{-1} \circ \mathbf{G}^{-1}) \circ (\mathbf{G} \circ \mathbf{F}) = \mathbf{I}$ and that is the end of the proof. □

4. Reduction of Hesitant Fuzzy Functions to Fuzzy Functions and Intuitionistic Fuzzy Functions

In this section, we explore how hesitant fuzzy functions (HFFs) serve as a generalization of both fuzzy functions (FFs) (Dib & Youssef 1991) and intuitionistic fuzzy functions (IFFs) (Fathi & Salleh 2008). By imposing certain structural constraints on the co-membership functions of HFFs, we can recover the definitions of FFs and IFFs. This reduction not only highlights the inclusiveness of HFFs within the broader fuzzy function framework but also provides a unified perspective for analyzing and comparing various types of fuzzy mappings.

To demonstrate how hesitant fuzzy functions (HFFs) generalize classical fuzzy functions (FFs) and intuitionistic fuzzy functions (IFFs) through structural reduction, we assume **F** is an HFF from X to Y with a co-membership function f_x for all $x \in X$. If each f_x contains a singleton in $[0,1]$, then HFF reduces to a classical fuzzy function from X to Y. This is because $\mathbf{F}(A) = \{\langle \mathcal{F}(x), f_x(h(x))\rangle | x \in X\} = \{\langle \mathcal{F}(x), \{a\}\rangle : x \in X, a \in [0,1]\} = B$ where B denotes a fuzzy set on Y. Thus, the hesitant nature disappears and the output becomes a single-valued fuzzy membership degree.

Lemma 4.1. Let $\mathbf{F} = \{\langle \mathcal{F}, f_x \rangle : x \in X\}$ be a hesitant fuzzy function from X to Y. If f_x is a singleton for any $x \in X$, then **F** is reduced to a fuzzy function.

Proof. Let $\mathbf{F}(A) = B$ for two HFSs A and B on X and Y respectively. Since $f_x(h(x)) = \{a\}$ for some $a \in [0,1]$, then we define the membership function of B as $\mu_B(x) = a$. Hence **F** is reduced to a fuzzy function. □

Let us consider the following example.

Example 4.1. According to Example 3.1.1 for $A = \{\langle 1, \{0.4, 0.5, 0.7\}\rangle, \langle 2, \{0.4, 0.7, 0.6\}\rangle, \langle 3, \{0.5, 0.6, 0.8\}\rangle\}$ be an HFS on X, let $\mathcal{F}(x) = 2x$ and $f_x(h(x)) = \{\max(h(x))\}$ where the hesitant fuzzy function from $X = \{1,2,3\}$ to Y is:

$$\begin{aligned}\mathbf{F}(A) &= \{\langle \mathcal{F}(x), f_x(h(x))\rangle | x \in X\} \\ &= \{\langle \mathcal{F}(1), f_1(\{0.4, 0.5, 0.7\})\rangle, \langle \mathcal{F}(2), f_2(\{0.4, 0.7, 0.6\})\rangle, \langle \mathcal{F}(3), f_3(\{0.5, 0.6, 0.8\})\rangle\} \\ &= \{\langle 2, \{0.7\}\rangle, \langle 4, \{0.7\}\rangle, \langle 6, \{0.8\}\rangle\} = B\end{aligned}$$

where B is an HFS on Y, we can also say B is an FS on Y.

Similarly, the definition of hesitant fuzzy functions (HFFs) can be reduced to the definition of intuitionistic fuzzy functions (IFFs) when we use specific functions for the co-membership function f_x such that the envelope of the hesitant fuzzy element of A, for all $x \in X$, reduces the HFS to IFS. These functions give us membership and non-membership degrees of membership in the image of HFE for each element. See the following example,

Let A be a hesitant fuzzy set on X, and let $h_A(x) \in P([0, 1])$ for each $x \in X$. Then the envelope of $h_A(x)$ of A, denoted $E_{env}(A)$, is defined as:

$$f_x(h(x)) = E_{env}(A) = \{\mu_A(x), v_A(x)\}$$

where $\mu_A(x) = \min(h_A(x))$ and $v_A(x) = 1 - \max(h_A(x))$ are the membership and non-membership degrees, respectively. Thus, each HFE is transformed into a pair of values satisfying the condition $\mu_A(x) + v_A(x) \leq 1$, and the output of the HFF becomes an intuitionistic fuzzy set.

Lemma 4.2. Let $\mathbf{F} = \{\langle \mathcal{F}, f_x \rangle : x \in X\}$ be a hesitant fuzzy function from X to Y, and A be an HFS on X. If for every $x \in X$, the co-membership function is defined as $f_x(h_A(x)) = \{\mu_A(x), v_A(x)\}$, then the function **F** reduces to an intuitionistic fuzzy function.

Proof. Let $\mathbf{F}(A) = B$ for two HFSs A and B on X and Y respectively. Since the image of each HFE $h(x)$ for all $x \in X$ under $f_x(h(x))$ is the pair $\{\mu_A(x), v_A(x)\}$, with $\mu_A(x), v_A(x) \in [0, 1]$ and $\mu_A(x) + v_A(x) \leq 1$, so the set B satisfies the definition of an IFS on Y. Therefore, **F** reduces to an Intuitionistic fuzzy function. □

Let us consider the following example.

Example 4.2. According to Example 3.1.1 for $A = \{\langle 1, \{0.4, 0.5, 0.7\}\rangle, \langle 2, \{0.4, 0.7, 0.6\}\rangle, \langle 3, \{0.5, 0.6, 0.8\}\rangle\}$ be an HFS on X, let $\mathcal{F}(x) = 2x$ and $f_x(h(x)) = E_{env}(h_A(x)) = (\mu_A(x), v_A(x))$ where $\mu_A(x) = h_A^-(x)$ and $v_A(x) = 1 - h_A^+(x)$. Therefore, the hesitant fuzzy function from $X = \{1, 2, 3\}$ to Y is:

$$
\begin{aligned}
\mathbf{F}(A) &= \{\langle \mathcal{F}(x), f_x(h(x))\rangle | x \in X\} \\
&= \{\langle \mathcal{F}(1), f_1(\{0.4, 0.5, 0.7\})\rangle, \langle \mathcal{F}(2), f_2(\{0.4, 0.7, 0.6\})\rangle, \langle \mathcal{F}(3), f_3(\{0.5, 0.6, 0.8\})\rangle\} \\
&= \{\langle 2, \{0.4, 1 - 0.7\}\rangle, \langle 4, \{0.4, 1 - 0.7\}\rangle, \langle 6, \{0.5, 1 - 0.8\}\rangle\} \\
&= \{\langle 2, \{0.4, 0.3\}\rangle, \langle 4, \{0.4, 0.3\}\rangle, \langle 6, \{0.5, 0.2\}\rangle\} = B.
\end{aligned}
$$

where B is an HFS on Y, we can also say B is an IFS on Y.

Discussion and potential implications: The results presented in this paper are not only of theoretical interest, but also provide insight into how hesitant information can be handled in functional settings. In particular, the image under an HFF helps describe how hesitant information is transferred from the domain to the codomain, while the preimage is relevant in reverse reasoning and decision-support contexts, where one seeks possible hesitant fuzzy inputs corresponding to a given hesitant fuzzy output. Moreover, composition is important for modelling multi-stage hesitant fuzzy functions, and invertibility may support the analysis or reconstruction of input-output relationships under hesitation. From this perspective, the proposed framework may offer a useful basis for future applications in decision-making, pattern recognition, and artificial intelligence systems involving vague or hesitant information. Table 1 provides a comparative summary of fuzzy functions, intuitionistic fuzzy functions, and hesitant fuzzy functions, highlighting the unifying role of the proposed framework.

Table 1 Comparison of fuzzy functions, intuitionistic fuzzy functions, and hesitant fuzzy functions

Aspect	Fuzzy Functions (Dib & Youssef 1991)	Intuitionistic Fuzzy Functions (Fathi & Salleh 2009)	Hesitant Fuzzy Functions (This work)
Underlying framework	Fuzzy set	Intuitionistic fuzzy set	Hesitant fuzzy set
Information assigned to each element	A single membership degree	A membership and non-membership pair	A finite set of possible membership degrees
Type of uncertainty represented	Fuzziness by one membership value	Fuzziness with explicit non-membership and indeterminacy	Hesitation among several possible membership values
Relation-based foundation	Based on the fuzzy Cartesian product and fuzzy relations	Based on the intuitionistic fuzzy Cartesian product and intuitionistic fuzzy relations	Based on the hesitant fuzzy Cartesian product and hesitant fuzzy relations
Functional interpretation as mapping	Maps a fuzzy set on X to a fuzzy set on Y	Maps an intuitionistic fuzzy set on X to an intuitionistic fuzzy set on Y	Maps a hesitant fuzzy set on X to a hesitant fuzzy set on Y
Image and preimage behavior	Defined in the fuzzy setting	Defined in the intuitionistic fuzzy setting	The image is unique, whereas the preimage generally corresponds to a class of possible hesitant fuzzy sets
Relation to the proposed framework	A reduced case under suitable constraints	A reduced case under suitable constraints	The general framework studied in this paper

5. Conclusion

This paper develops a formal theoretical framework for hesitant fuzzy functions (HFFs) within the setting of hesitant fuzzy relations. Specifically, hesitant fuzzy functions are defined as a special class of hesitant fuzzy relations satisfying four essential conditions: existence, uniqueness, order preservation, and boundary preservation. The paper examines several fundamental properties of HFFs, including equality, image and preimage behavior, injectivity, surjectivity, composition, and invertibility, and provides illustrative examples and counterexamples to clarify these notions in the hesitant fuzzy setting. A central contribution of the paper is the formal treatment of image and preimage under hesitant fuzzy functions. In particular, the image of a hesitant fuzzy set under an HFF is uniquely determined, whereas the preimage generally corresponds to a class of possible hesitant fuzzy sets rather than a single set. These results provide a clearer and more consistent interpretation of hesitant fuzzy mappings. In addition, the paper establishes several basic theorems that support the structural behavior of HFFs and shows that the proposed framework admits natural reductions to fuzzy functions and intuitionistic fuzzy functions under suitable structural constraints on the co-membership functions of HFFs. This highlights the unifying role of HFFs as a general framework for representing hesitation in functional settings. As a possible direction for future research, the present framework may be extended to hesitant fuzzy algebra through appropriate hesitant fuzzy spaces and hesitant fuzzy binary operations. This framework has potential applications in pattern recognition, complex decision-making systems, and artificial intelligence dealing with vague or imprecise information.

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Department of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 UKM Bangi
Selangor, MALAYSIA.
E-mail: p126635@siswa.ukm.edu.my, syahida@ukm.edu.my[*]

Department of Mathematics
Faculty of Science
Ajloun National University
P.O. Box 43, Ajloun 26810
Ajloun, JORDAN.
E-mail: alkouriabdulazeez@anu.edu.jo

Department of Mathematics
Faculty of Arts and Science
Amman Arab University
P.O. Box 2234 - Amman 11953
Amman, JORDAN.
E-mail: e.hussein@aau.edu.jo

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*Corresponding author