Journal of Quality Measurement and Analysis *JQMA* 22(SI) 2026, 163-179
e-ISSN: 2600-8602 *http://www.ukm.my/jqma*
https://doi.org/10.17576/jqma.22si.2026.10

THIRD HANKEL DETERMINANT FOR CERTAIN SUBCLASSES OF UNIVALENT FUNCTION ASSOCIATED WITH SAKAGUCHI TYPE FUNCTIONS

(*Penentu Hankel Ketiga bagi Subkelas Fungsi Univalen Tertentu Bersekutu dengan Fungsi Jenis Sakaguchi*)

LEE CHEW YEE* & MASLINA DARUS

ABSTRACT

In this article, a subclass of univalent function associated with generalised Sakaguchi function is defined. The aim of the study is to investigate some coefficient bounds and inequalities of certain determinants. In addition, we provide some insights into the geometrical and analytical properties, by investigating the bounds of the third Hankel determinants.

Keywords: Hankel determinants; Sakaguchi type functions; univalent functions

ABSTRAK

Dalam artikel ini, suatu subkelas fungsi univalen berkaitan dengan fungsi berbentuk Sakaguchi teritlak adalah ditakrif. Tujuan kajian ini adalah untuk mendapatkan batas atas bagi beberapa pekali awal dan selanjutnya, memberi beberapa ketaksamaan penentu bagi kelas fungsi yang diperkenalkan. Malah, sifat geometri dan analisis bagi kelas ini diperhalusi dengan memperoleh pekali Hankel ketiga.

Keywords: penentu Hankel; fungsi Sakaguchi; fungsi univalent

1. Introduction

Let $\mathcal{A}$ signify the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ expressed in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and normalized with $f(0) = 0$ and $f'(0) = 1$. A subclass of $\mathcal{A}$ consisting of univalent functions is represented by $\mathcal{S}$. Let $\mathcal{S}^*$ and $\mathcal{K}$ denotes the subclasses of $\mathcal{S}$, containing the set of starlike and convex functions, respectively, in which

$$\mathcal{S}^* = \left\{f \in \mathcal{S}: Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in \mathbb{U}\right\}$$

and

$$\mathcal{K} = \left\{f \in \mathcal{S}: Re\left(\frac{zf''(z)}{f'(z)}\right) > 0, z \in \mathbb{U}\right\}.$$

A subclass of $\mathcal{S}$ containing functions p which map $\mathbb{U}$ onto the right half complex plane, and satisfy $Re\left(p(z)\right) > 0, p(0) = 1$ is denoted by $\mathcal{P}$.

Sakaguchi (1959) established a subclass of $\mathcal{S}$ which contains functions satisfying the inequality

$$Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, z \in \mathbb{U}.$$

These functions are starlike about symmetric points.

Similarly, Das and Singh (1977) introduced another subclass of $\mathcal{S}$ which are convex about symmetric points. These functions satisfy the inequality

$$Re\left(\frac{(zf'(z))'}{\left(f(z) - f(-z)\right)'}\right) > 0, z \in \mathbb{U}.$$

Combining the previous two subclasses, Selvaraj and Vasanthi (2011) introduced a class of α-symmetric functions with respect to symmetric points, which satisfy the inequality

$$Re\left(\frac{2zf'(z) + 2\alpha z^2 f''(z)}{(1-\alpha)\left(f(z) - f(-z)\right) + \alpha z\left(f(z) - f(-z)\right)'}\right) > 0, \alpha \in [0,1], z \in \mathbb{U}.$$

Singh *et al*. (2019) further generalise the class, by introducing a class of functions which satisfy the quasi-subordination and biunivalent conditions

$$\frac{1}{\gamma}\left(\frac{2zf'(z) + 2\alpha z^2 f''(z)}{(1-\alpha)\left(f(z) - f(-z)\right) + \alpha z\left(f(z) - f(-z)\right)'} - 1\right) \prec_q (\phi(z) - 1)$$

and

$$\frac{1}{\gamma}\left(\frac{2wg'(w) + 2\alpha w^2 g''(w)}{(1-\alpha)\left(g(w) - g(-w)\right) + \alpha w\left(g(w) - g(-w)\right)'} - 1\right) \prec_q (\phi(w) - 1)$$

where $g = f^{-1}$, $z, w \in \mathbb{U}$ and ϕ is an analytic function in $\mathbb{U}$ with positive real second coefficient.

Inspired by Singh *et al*. (2019), however by only considering the univalent case, we define a class of functions associated with generalised Sakaguchi type function as follows,

Definition 1.1. A function $f \in \mathcal{S}$ is presumed to be in class $\mathcal{S}^*_{S,\bar{\alpha},\gamma,\Xi}$ if for all $z \in \mathbb{U}$, $0 \leq \bar{\alpha} \leq 1$ and $0 \leq \Xi < 1$, the following conditions hold,

$$Re\left\{1 + \frac{1}{\gamma}\left[\frac{2zf'(z) + 2\bar{\alpha}\, z^2 f''(z)}{(1-\bar{\alpha})\left(f(z) - f(-z)\right) + \bar{\alpha} z\left(f(z) - f(-z)\right)'} - 1\right]\right\} > \Xi.$$

Remark 1.2. *When $\gamma = 1$, $\bar{\alpha} = 0$ and $\Xi = 0$, $\mathcal{S}^*_{S,0,1,0}$ is affiliated with the class introduced by Sakaguchi (1959).*

Remark 1.3. *When $\gamma = 1$, $\bar{\alpha} = 1$ and $\Xi = 0$, $\mathcal{S}^*_{S,1,1,0}$ is affiliated with the class introduced by Das and Singh (1977).*

Remark 1.4. *When $\gamma = 1$, $\bar{\alpha} = \alpha$ and $\Xi = 0$, $\mathcal{S}^*_{S,\alpha,1,0}$ is the class introduced by Selvaraj and Vasanthi (2011).*

As studied by Noonan and Thomas (1976), q^{th} Hankel determinants, $H_q(n)$ of analytic functions of the form (1), where $q \geq 1$ and $n \geq 1$, $q, n \in \mathbb{N}$, is defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

with $a_1 = 1$. To be precise, we will be considering

$$H_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

The concept of Hankel determinant is competent in the study of power series with integral coefficients. For specific values of q and n, many authors have investigated the rate of growth of $H_q(n)$ as $n \to \infty$ and evaluated the bounds on it. For instance, Pommerenke (1966) showed that the Hankel determinants of univalent functions satisfy $|H_q(n)| < kn^{-\left(\frac{1}{2}+\beta\right)q+\frac{3}{2}}$, where k depends only on q and $\beta > \frac{1}{4000}$. Hayman (1968) later illustrated that for areally mean univalent functions, the Hankel determinant satisfy $|H_q(n)| < C\sqrt{n}$ (C is an absolute constant). It should be notable that the Hankel determinant $|H_2(1)| = |a_3 - a_2^2|$ is also recognised as Fekete-Szegö functional (Fekete & Szegö 1933). This functional is also generalised as $|a_3 - \mu a_2^2|$, in which μ is a parameter satisfying $0 \leq \mu < 1$. Sharp upper bounds on $H_2(2) = |a_2a_4 - a_3^2|$ was studied by Janteng *et al.* (2006) for functions whose derivative has positive real part. Bounds for $H_3(1)$ have also been inspected by several authors, such as Babalola (2007) , Raza and Malik (2013), Bansal *et al.* (2015) and Mishra *et al.* (2016). Later, Vanitha *et al.* (2018) investigated the third Hankel determinant of some class of univalent functions related with generalised Sakaguchi functions. However, this generalisation is different from which is defined in Definition 1.1. In fact, there is no known work done on the third Hankel determinant using the said definition. Thus, in this paper, we are tackling this problem and deriving various determinants related to it.

Third Hankel determinant involves the first five coefficients of Taylor series, as compared to second Hankel determinant which involves only three. This captures deeper insights into geometric behavior of complex functions and provides a more comprehensive analysis of their structure. However, higher order determinant is not considered since the computation is too complex.

2. Preliminary Results

Suppose that $\mathcal{P}$ denotes the class of functions

$$p(z) = 1 + \sum_{i=1}^{\infty} p_i z^i$$

which satisfy $Re\left(p(z)\right) > 0, z \in \mathbb{U}$.

Lemma 2.1. (Duren 1983) *Suppose that* $p \in \mathcal{P}$, *then*

$$|p_n| \leq 2, n \in 1, 2, \dots .$$

This inequality is sharp.

Lemma 2.2. (Libera & Złotkiewicz 1982; 1983) *Suppose that* $p \in \mathcal{P}$. *Then there exists* z *and a complex parameter* x *with* $|z| < 1$ *and* $|x| < 1$ *such that*

$$2p_2 = p_1^2 + x(4 - p_1^2)$$

and

$$4p_3 = p_1^3 + (4 - p_1^2)[2xp_1 - x^2 p_1 + 2z(1 - |x|^2)].$$

Lemma 2.3. (Keogh & Merkes 1969) *Suppose that* $p \in \mathcal{P}$. *Then*

$$\left|p_2 - \frac{vp_1^2}{2}\right| = \begin{cases} 2(1 - v) & if\ v \leq 0, \\ 2 & if\ 0 \leq v \leq 2, \\ 2(v - 1) & if\ v \geq 2. \end{cases}$$

3. Main Results

In this section, we investigate the coefficient bounds and perform some determinant calculations. Then, the third Hankel determinant is evaluated.

Theorem 3.1. *Suppose that* f *belongs to the class* $\mathcal{S}^*_{S,\bar{\alpha},\gamma,\Xi}$, *then*

$$|a_2| \leq \frac{(1 - \Xi)|\gamma|}{1 + \bar{\alpha}},$$
$$|a_3| \leq \frac{(1 - \Xi)\ |\gamma|}{1 + 2\bar{\alpha}},$$
$$|a_4| \leq \frac{(1 - \Xi)|\gamma|\ [1 + (1 - \Xi)|\gamma|]}{2 + 6\bar{\alpha}},$$

and

$$|a_5| \leq \frac{(1 - \Xi)|\gamma|[1 + (1 - \Xi)|\gamma|]}{2 + 8\bar{\alpha}}.$$

Proof. Let $f \in \mathcal{S}^*_{S,\bar{\alpha},\gamma,\Xi}$, then according to Definition 1.1, for all $z \in \mathbb{U}$, we have

$$1 + \frac{1}{\gamma}\left[\frac{2zf'(z) + 2\bar{\alpha}\ z^2 f''(z)}{(1 - \bar{\alpha})\big(f(z) - f(-z)\big) + \bar{\alpha} z\big(f(z) - f(-z)\big)'} - 1\right] = \Xi + (1 - \Xi)\ p(z), \qquad (2)$$

where

$$\Xi + (1 - \Xi)\, p(z) = 1 + \sum_{n=1}^{\infty} (1 - \Xi)\, p_n z^n.$$

By comparing coefficients, we have

$$\frac{2a_2(1+\bar{\alpha})}{\gamma} = (1-\Xi)p_1,$$
$$\frac{2a_3(1+2\bar{\alpha})}{\gamma} = (1-\Xi)p_2,$$

$$\frac{(4+12\bar{\alpha})a_4 - (2+2\bar{\alpha})(1+2\bar{\alpha})a_2a_3}{\gamma} = (1-\Xi)p_3$$

and

$$\frac{(4+16\bar{\alpha})a_5 - (2+4\bar{\alpha})(1+2\bar{\alpha})a_3^2}{\gamma} = (1-\Xi)p_4$$

which can be transformed into

$$a_2 = \frac{(1-\Xi)\gamma p_1}{2+2\bar{\alpha}}, \tag{3}$$
$$a_3 = \frac{(1-\Xi)\gamma p_2}{2+4\bar{\alpha}}, \tag{4}$$
$$a_4 = \frac{(1-\Xi)[2\gamma p_3 + (1-\Xi)\gamma^2 p_1 p_2]}{(8+24\bar{\alpha})}, \tag{5}$$

and

$$a_5 = \frac{(1-\Xi)[2\gamma p_4 + (1-\Xi)\gamma^2 p_2^2]}{(8+32\bar{\alpha})}. \tag{6}$$

Using Lemma 2.1 and triangle inequality, we achieve

$$|a_2| \leq \frac{(1-\Xi)|\gamma|}{1+\bar{\alpha}},$$
$$|a_3| \leq \frac{(1-\Xi)\,|\gamma|}{1+2\bar{\alpha}},$$
$$|a_4| \leq \frac{(1-\Xi)|\gamma|\,[1+(1-\Xi)|\gamma|]}{2+6\bar{\alpha}},$$

and

$$|a_5| \leq \frac{(1-\Xi)|\gamma|[1+(1-\Xi)|\gamma|]}{(2+8\bar{\alpha})}. \quad \square$$

For the functions defined by (2), equality is attainable by taking $p_n = 2$ in Eqs. (3), (4), (5) and (6).

Remark 3.2. *The bounds for Theorem 3.1 is sharp when $\Xi = 0 = \bar{\alpha}$. In this case, we have*

$$|a_2| \leq |\gamma|,$$

$$|a_3| \le |\gamma|,$$
$$|a_4| \le \frac{|\gamma| \ + |\gamma|^2}{2},$$

and

$$|a_5| \le \frac{|\gamma| + |\gamma|^2}{2}.$$

Theorem 3.3. *Suppose that f belongs to the class $\mathcal{S}^*_{S,\bar{\alpha},\gamma,\Xi}$, then*

$$\left|a_2^2 - a_3\right| \le \frac{(1-\Xi)^2\,|\gamma|^2}{(1+\bar{\alpha})^2} + \frac{(1-\Xi)|\gamma|}{1+2\bar{\alpha}}.$$

Proof. From Eqs. (3) and (4), we have

$$\begin{aligned} a_2^2 - a_3 \ &= \left(\frac{(1-\Xi)\gamma p_1}{2+2\bar{\alpha}}\right)^2 - \frac{(1-\Xi)\gamma p_2}{2+4\bar{\alpha}} \\ &= \left[\frac{(1-\Xi)\gamma}{2+4\bar{\alpha}}\right]\left[\frac{(1-\Xi)(2+4\bar{\alpha})\gamma p_1^2}{(2+2\bar{\alpha})^2} - p_2\right]. \end{aligned} \tag{7}$$

It follows from Lemma 2.3, and the fact that $\frac{(1-\Xi)(2+4\bar{\alpha})}{(2+2\bar{\alpha})^2} \ge 0$, we have

$$\begin{aligned} &\left|a_2^2 - a_3\right| \\ &= \begin{cases} \dfrac{(1-\Xi)|\gamma|}{1+2\bar{\alpha}}, & \text{if } 0 < \dfrac{2(1-\Xi)(2+4\bar{\alpha})\,|\gamma|}{(2+2\bar{\alpha})^2} \le 2 \\ \dfrac{(1-\Xi)|\gamma|}{1+2\bar{\alpha}}\left|\dfrac{(1-\Xi)(1+2\bar{\alpha})\,|\gamma|}{(1+\bar{\alpha})^2} - 1\right|, & \text{if } \dfrac{2(1-\Xi)(2+4\bar{\alpha})\,|\gamma|}{(2+2\bar{\alpha})^2} \ge 2 \end{cases} \\ &= \begin{cases} \dfrac{(1-\Xi)|\gamma|}{1+2\bar{\alpha}}, & \text{if } 0 < \dfrac{(1-\Xi)(2+4\bar{\alpha})\,|\gamma|}{(2+2\bar{\alpha})^2} \le 1 \\ \left|\dfrac{(1-\Xi)^2\,|\gamma|^2}{(1+\bar{\alpha})^2} - \dfrac{(1-\Xi)|\gamma|}{1+2\bar{\alpha}}\right|, & \text{if } \dfrac{(1-\Xi)(2+4\bar{\alpha})\,|\gamma|}{(2+2\bar{\alpha})^2} \ge 1. \end{cases} \end{aligned}$$

Using triangle inequality, we have

$$\left|a_2^2 - a_3\right| \le \begin{cases} \dfrac{(1-\Xi)|\gamma|}{1+2\bar{\alpha}}, & \text{if } 0 < \dfrac{(1-\Xi)(2+4\bar{\alpha})\,|\gamma|}{(2+2\bar{\alpha})^2} \le 1 \\ \dfrac{(1-\Xi)^2\,|\gamma|^2}{(1+\bar{\alpha})^2} + \dfrac{(1-\Xi)|\gamma|}{1+2\bar{\alpha}}, & \text{if } \dfrac{(1-\Xi)(2+4\bar{\alpha})\,|\gamma|}{(2+2\bar{\alpha})^2} \ge 1. \end{cases}$$

Obviously

$$\frac{(1-\Xi)^2\,|\gamma|^2}{(1+\bar{\alpha})^2} + \frac{(1-\Xi)|\gamma|}{1+2\bar{\alpha}} \ge \frac{(1-\Xi)|\gamma|}{(1+2\bar{\alpha})}.$$

Let the expression on the left be $F(|\gamma|)$, and take its first derivative, we have

$$\frac{d\,F(|\gamma|)}{d|\gamma|}=\frac{d}{d|\gamma|}\left[\frac{(1-\Xi)^2\,|\gamma|^2}{(1+\bar{\alpha})^2}+\frac{(1-\Xi)|\gamma|}{1+2\bar{\alpha}}\right]=\frac{2(1-\Xi)^2\,|\gamma|}{(1+\bar{\alpha})^2}+\frac{(1-\Xi)}{1+2\bar{\alpha}}>0.$$

So it is an increasing function with respect to $|\gamma|$. Hence we obtain

$$\left|a_2^2-a_3\right|\leq\frac{(1-\Xi)^2\,|\gamma|^2}{(1+\bar{\alpha})^2}+\frac{(1-\Xi)|\gamma|}{1+2\bar{\alpha}}.\quad\square$$

For the functions defined by (2), equality can be attained in (7) by taking $p_1=2$ and $p_2=-2$.

Remark 3.4. *By elementary analysis,* $F(|\gamma|)$ *attain its maximun when* $\Xi=\bar{\alpha}=0$, *thus Theorem 3.3 reduces to* $\left|a_2^2-a_3\right|\leq|\gamma|^2+|\gamma|$.

Theorem 3.5. *Let* f *be a function in the class* $\mathcal{S}^*_{S,\bar{\alpha},\gamma,\Xi}$, *then*

$$|a_2a_3-a_4|\leq\begin{cases}\dfrac{(1-\Xi)|\gamma|}{8(1+3\bar{\alpha})}(-\rho^3-\rho^2+6\rho+4)+\dfrac{(1-\Xi)^2|\gamma|^2\,(1+3\bar{\alpha}-2\bar{\alpha}^2)}{4(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}\rho, & when\ \rho<2\\[2ex] \dfrac{(1-\Xi)|\gamma|}{2(1+3\bar{\alpha})}+\dfrac{(1-\Xi)^2|\gamma|^2\,(1+3\bar{\alpha}-2\bar{\alpha}^2)}{2(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}, & when\ \rho\geq 2\end{cases},$$

where

$$\rho=-\frac{1}{3}+\sqrt{\frac{6(1-\Xi)|\gamma|(1+3\bar{\alpha}-2\bar{\alpha}^2)+19(1+2\bar{\alpha})(1+\bar{\alpha})}{9(1+2\bar{\alpha})(1+\bar{\alpha})}}.$$

Proof. From Eqs. (3), (4) and (5), we have

$$\begin{aligned}a_2a_3-a_4&=\left(\frac{(1-\Xi)\gamma p_1}{2+2\bar{\alpha}}\right)\left(\frac{(1-\Xi)\gamma p_2}{2+4\bar{\alpha}}\right)-\frac{(1-\Xi)[2\gamma p_3+(1-\Xi)\gamma^2p_1p_2]}{(8+24\bar{\alpha})}\\&=\frac{(1-\Xi)\gamma}{(2+2\bar{\alpha})}\left[\left(\frac{(1-\Xi)\gamma p_1p_2}{2+4\bar{\alpha}}\right)-\frac{[(2+2\bar{\alpha})p_3+(1-\Xi)(1+\bar{\alpha})\gamma p_1p_2]}{(4+12\bar{\alpha})}\right]\\&=\frac{(1-\Xi)\gamma}{(2+2\bar{\alpha})(4+12\bar{\alpha})}\left[(1-\Xi)\gamma p_1p_2\frac{1+3\bar{\alpha}-2\bar{\alpha}^2}{(1+2\bar{\alpha})}-(2+2\bar{\alpha})p_3\right].\end{aligned}\tag{8}$$

Using Lemma 2.2, we obtain

$$|a_2a_3-a_4|$$

$$= \frac{(1-\Xi)|\gamma|}{(2+2\bar{\alpha})(4+12\bar{\alpha})}\left|\frac{(1-\Xi)(1+3\bar{\alpha}-2\bar{\alpha}^2)\gamma p_1\left(p_1^2+x(4-p_1^2)\right)}{2(1+2\bar{\alpha})}-\frac{(2+2\bar{\alpha})p_1^3}{4}\right.$$
$$\left.-\frac{(2+2\bar{\alpha})(4-p_1^2)[2xp_1-x^2p_1+2z(1-|x|^2)]}{4}\right|$$
$$= \frac{(1-\Xi)|\gamma|}{(2+2\bar{\alpha})(4+12\bar{\alpha})}\left|\frac{(1-\Xi)(1+3\bar{\alpha}-2\bar{\alpha}^2)\gamma p_1^3}{2(1+2\bar{\alpha})}\right.$$
$$+\frac{(1-\Xi)(1+3\bar{\alpha}-2\bar{\alpha}^2)\gamma p_1 x(4-p_1^2)}{2(1+2\bar{\alpha})}-\frac{(1+\bar{\alpha})p_1^3}{2}$$
$$\left.-\frac{(1+\bar{\alpha})(4-p_1^2)[2xp_1-x^2p_1+2z(1-|x|^2)]}{2}\right|$$
$$= \frac{(1-\Xi)|\gamma|}{16(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}\big|[(1-\Xi)(1+3\bar{\alpha}-2\bar{\alpha}^2)\gamma-(1+\bar{\alpha})(1+2\bar{\alpha})]p_1^3$$
$$+xp_1(4-p_1^2)[(1-\Xi)(1+3\bar{\alpha}-2\bar{\alpha}^2)\gamma-2(1+\bar{\alpha})(1+2\bar{\alpha})]$$
$$+x^2p_1(4-p_1^2)(1+\bar{\alpha})(1+2\bar{\alpha})$$
$$-2z(1-|x|^2)(4-p_1^2)(1+\bar{\alpha})(1+2\bar{\alpha})\big|.$$

Apply triangle inequality, and use the assumption that $|z|<1$, $1-|x|^2\le 1$, we obtain

$$|a_2a_3-a_4|$$
$$\le \frac{(1-\Xi)|\gamma|}{16(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}\{[(1-\Xi)(1+3\bar{\alpha}-2\bar{\alpha}^2)\,|\gamma|+(1+\bar{\alpha})(1+2\bar{\alpha})]p_1^3$$
$$+|x|p_1(4-p_1^2)[(1-\Xi)(1+3\bar{\alpha}-2\bar{\alpha}^2)|\gamma|+2(1+\bar{\alpha})(1+2\bar{\alpha})]$$
$$+|x|^2p_1(4-p_1^2)(1+\bar{\alpha})(1+2\bar{\alpha})+2(4-p_1^2)(1+\bar{\alpha})(1+2\bar{\alpha})\}.$$

Let the right side of inequality be $G(|x|,p_1)$, and differentiate G with respect to $|x|$,

$$\frac{\partial\, G(|x|,p_1)}{\partial|x|}$$
$$= \frac{(1-\Xi)|\gamma|}{16(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}[p_1(4-p_1^2)\,[(1-\Xi)(1+3\bar{\alpha}-2\bar{\alpha}^2)|\gamma|$$
$$+2(1+\bar{\alpha})(1+2\bar{\alpha})]+2\,|x|p_1(4-p_1^2)(1+\bar{\alpha})(1+2\bar{\alpha})]$$

which is always positive, which means that G is an increasing function of $|x|$. So maximum occurs when $|x|=1$, and we have

$$G(1,p_1)$$
$$= \frac{(1-\Xi)|\gamma|}{16(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}\{[(1-\Xi)(1+3\bar{\alpha}-2\bar{\alpha}^2)\,|\gamma|+(1+\bar{\alpha})(1+2\bar{\alpha})]p_1^3$$
$$+p_1(4-p_1^2)[(1-\beta)(1+3\bar{\alpha}-2\bar{\alpha}^2)|\gamma|+2(1+\bar{\alpha})(1+2\bar{\alpha})]$$
$$+p_1(4-p_1^2)(1+\bar{\alpha})(1+2\bar{\alpha})+2(4-p_1^2)(1+\bar{\alpha})(1+2\bar{\alpha})\}$$
$$= \frac{(1-\Xi)|\gamma|}{16(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}\{-2(1+\bar{\alpha})(1+2\bar{\alpha})p_1^3-2(1+\bar{\alpha})(1+2\bar{\alpha})p_1^2$$
$$+4[(1-\Xi)(1+3\bar{\alpha}-2\bar{\alpha}^2)|\gamma|+3(1+\bar{\alpha})(1+2\bar{\alpha})]p_1$$
$$+8(1+\bar{\alpha})(1+2\bar{\alpha})\}$$

$$
\begin{aligned}
&= \frac{(1-\Xi)|\gamma|}{8(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}\{-(1+\bar{\alpha})(1+2\bar{\alpha})p_1^3 - (1+\bar{\alpha})(1+2\bar{\alpha})p_1^2 \\
&\qquad + 2[(1-\Xi)(1+3\bar{\alpha}-2\bar{\alpha}^2)|\gamma| + 3(1+\bar{\alpha})(1+2\bar{\alpha})]p_1 \\
&\qquad + 4(1+\bar{\alpha})(1+2\bar{\alpha})\} \\
&= \frac{(1-\Xi)|\gamma|}{8(1+3\bar{\alpha})}(-p_1^3 - p_1^2 + 6p_1 + 4) + \frac{(1-\Xi)^2|\gamma|^2\,(1+3\bar{\alpha}-2\bar{\alpha}^2)}{4(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}p_1.
\end{aligned}
$$

Differentiate $G(1, p_1)$ with respect to p_1,

$$
\frac{\partial\, G(1,p_1)}{\partial p_1} = \frac{(1-\Xi)|\gamma|}{8(1+3\bar{\alpha})}(-3p_1^2 - 2p_1 + 6) + \frac{(1-\Xi)^2|\gamma|^2\,(1+3\bar{\alpha}-2\bar{\alpha}^2)}{4(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}.
$$

Since function $y = -3p_1^2 - 2p_1 + 6$ has roots $p_1 = \frac{-1\pm\sqrt{19}}{3}$ and is concaving downwards, it is positive in the interval $p_1 \in \left(\frac{-1-\sqrt{19}}{3}, \frac{-1+\sqrt{19}}{3}\right)$. Therefore $G(1, p_1)$ is always increasing when $p_1 \in \left[0, \frac{-1+\sqrt{19}}{3}\right]$.

When $p_1 \in \left[\frac{-1+\sqrt{19}}{3}, 2\right]$ and $G(1, p_1)$ is concaving downwards, maximum occurs when

$$
\begin{aligned}
&\frac{\partial\, G(1,p_1)}{\partial p_1} = \frac{(1-\Xi)|\gamma|}{8(1+3\bar{\alpha})}(-3p_1^2 - 2p_1 + 6) + \frac{(1-\Xi)^2|\gamma|^2\,(1+3\bar{\alpha}-2\bar{\alpha}^2)}{4(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})} = 0 \\
&\frac{2(1-\Xi)|\gamma|(1+3\bar{\alpha}-2\bar{\alpha}^2)}{(1+2\bar{\alpha})(1+\bar{\alpha})} = 3p_1^2 + 2p_1 - 6
\end{aligned}
$$

By completing the squares,

$$
\begin{aligned}
\frac{2(1-\Xi)|\gamma|(1+3\bar{\alpha}-2\bar{\alpha}^2)}{3(1+2\bar{\alpha})(1+\bar{\alpha})} + \frac{19}{9} &= p_1^2 + \frac{2}{3}p_1 + \frac{1}{9} \\
\frac{6(1-\Xi)|\gamma|(1+3\bar{\alpha}-2\bar{\alpha}^2) + 19(1+2\bar{\alpha})(1+\bar{\alpha})}{9(1+2\bar{\alpha})(1+\bar{\alpha})} &= \left(p_1 + \frac{1}{3}\right)^2,
\end{aligned}
$$

hence

$$
p_1 = -\frac{1}{3} - \sqrt{\frac{6(1-\Xi)|\gamma|(1+3\bar{\alpha}-2\bar{\alpha}^2) + 19(1+2\bar{\alpha})(1+\bar{\alpha})}{9(1+2\bar{\alpha})(1+\bar{\alpha})}} \notin [0,2]
$$

or

$$
p_1 = -\frac{1}{3} + \sqrt{\frac{6(1-\Xi)|\gamma|(1+3\bar{\alpha}-2\bar{\alpha}^2) + 19(1+2\bar{\alpha})(1+\bar{\alpha})}{9(1+2\bar{\alpha})(1+\bar{\alpha})}} = \rho.
$$

Therefore, by assuming ρ as the value of p_1 which maximise $G(1, p_1)$, we have

$\max G(1, p_1)$

$$
\begin{aligned}
&= G(1,\rho)\\
&= \frac{(1-\Xi)|\gamma|}{8(1+3\bar{\alpha})}(-\rho^3-\rho^2+6\rho+4)+\frac{(1-\Xi)^2|\gamma|^2\,(1+3\bar{\alpha}-2\bar{\alpha}^2)}{4(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}\rho.
\end{aligned}
$$

When $p_1 \in \left[\frac{-1+\sqrt{19}}{3}, 2\right]$ and $G(1,p_1)$ is still increasing ($\rho \geq 2$), maximum occurs when $p_1 = 2$, so we take

$$
\begin{aligned}
&\max G(1,p_1)\\
&= G(1,2)\\
&= \frac{(1-\Xi)|\gamma|}{8(1+3\bar{\alpha})}(4)+\frac{(1-\Xi)^2|\gamma|^2\,(1+3\bar{\alpha}-2\bar{\alpha}^2)}{4(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}(2)\\
&= \frac{(1-\Xi)|\gamma|}{2(1+3\bar{\alpha})}+\frac{(1-\Xi)^2|\gamma|^2\,(1+3\bar{\alpha}-2\bar{\alpha}^2)}{2(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&|a_2a_3-a_4|\\
&\leq \begin{cases}
\dfrac{(1-\Xi)|\gamma|}{8(1+3\bar{\alpha})}(-\rho^3-\rho^2+6\rho+4)+\dfrac{(1-\Xi)^2|\gamma|^2\,(1+3\bar{\alpha}-2\bar{\alpha}^2)}{4(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}\rho, & \text{when } \rho < 2\\
\dfrac{(1-\Xi)|\gamma|}{2(1+3\bar{\alpha})}+\dfrac{(1-\Xi)^2|\gamma|^2\,(1+3\bar{\alpha}-2\bar{\alpha}^2)}{2(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}, & \text{when } \rho \geq 2
\end{cases},
\end{aligned}
$$

where

$$
\rho = -\frac{1}{3}+\sqrt{\frac{6(1-\Xi)|\gamma|(1+3\bar{\alpha}-2\bar{\alpha}^2)+19(1+2\bar{\alpha})(1+\bar{\alpha})}{9(1+2\bar{\alpha})(1+\bar{\alpha})}}. \quad \square
$$

For the functions defined by (2), equality can be attained in (8) by taking $p_1 = p_2 = 2$ and $p_3 = -2$.

Theorem 3.6. *Suppose that f belongs to the class $\mathcal{S}^*_{S,\bar{\alpha},\gamma,\Xi}$, then*

$$
|a_2a_4-a_3^2| \leq \frac{(1-\Xi)^2|\gamma|^2}{4(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{32\kappa-63+\frac{188\bar{\alpha}^2+252\bar{\alpha}+63}{(1+2\bar{\alpha})^2}\right\},
$$

where

$$
\kappa = 3+(1-\Xi)\,|\gamma|.
$$

Proof. From (3), (4) and (5), we have

$$
\begin{aligned}
&a_2a_4-a_3^2\\
&= \left(\frac{(1-\Xi)\gamma p_1}{2+2\bar{\alpha}}\right)\frac{(1-\Xi)[2\gamma p_3+(1-\Xi)\gamma^2p_1p_2]}{(8+24\bar{\alpha})}-\left(\frac{(1-\Xi)\gamma p_2}{2+4\bar{\alpha}}\right)^2
\end{aligned}
$$

$$= (1-\Xi)^2|\gamma|^2\left[\frac{2p_1p_3+(1-\Xi)\gamma p_1^2p_2}{(2+2\bar{\alpha})(8+24\bar{\alpha})}-\frac{p_2^2}{(2+4\bar{\alpha})^2}\right]. \quad (9)$$

Using Lemma 2.2, we obtain

$$\begin{aligned}
&\left|a_2a_4-a_3^2\right| \\
&= (1-\Xi)^2|\gamma|^2\left|\frac{p_1\{p_1^3+(4-p_1^2)[2xp_1-x^2p_1+2z(1-|x|^2)]\}}{2(2+2\bar{\alpha})(8+24\bar{\alpha})}\right. \\
&\qquad +\frac{(1-\Xi)\gamma p_1^2[p_1^2+x(4-p_1^2)]}{2(2+2\bar{\alpha})(8+24\bar{\alpha})} \\
&\qquad \left.-\frac{p_1^4+x^2(4-p_1^2)^2+2xp_1^2(4-p_1^2)}{4(2+4\bar{\alpha})^2}\right| \\
&= (1-\Xi)^2|\gamma|^2\left|\frac{p_1^4+2xp_1^2(4-p_1^2)-x^2p_1^2(4-p_1^2)+2z(1-|x|^2)p_1(4-p_1^2)}{2(2+2\bar{\alpha})(8+24\bar{\alpha})}\right. \\
&\qquad +\frac{(1-\Xi)\gamma p_1^4+(1-\Xi)\gamma p_1^2\,x(4-p_1^2)}{2(2+2\bar{\alpha})(8+24\bar{\alpha})} \\
&\qquad \left.-\frac{p_1^4+x^2(16+p_1^4-8p_1^2)+2xp_1^2(4-p_1^2)}{4(2+4\bar{\alpha})^2}\right| \\
&= (1-\Xi)^2|\gamma|^2\left|\left[\frac{1+(1-\Xi)\gamma}{2(2+2\bar{\alpha})(8+24\bar{\alpha})}-\frac{1}{4(2+4\bar{\alpha})^2}\right]p_1^4\right. \\
&\qquad +\left[\frac{2+(1-\Xi)\gamma}{2(2+2\bar{\alpha})(8+24\bar{\alpha})}-\frac{2}{4(2+4\bar{\alpha})^2}\right]xp_1^2(4-p_1^2) \\
&\qquad -\frac{x^2p_1^2(4-p_1^2)}{2(2+2\bar{\alpha})(8+24\bar{\alpha})}+\frac{2z(1-|x|^2)p_1(4-p_1^2)}{2(2+2\bar{\alpha})(8+24\bar{\alpha})} \\
&\qquad \left.-\frac{x^2(16+p_1^4-8p_1^2)}{4(2+4\bar{\alpha})^2}\right|.
\end{aligned}$$

Apply triangle inequality, and since $|z|<1$, $1-|x|^2\le 1$, we obtain

$$\begin{aligned}
&\left|a_2a_4-a_3^2\right| \\
&\le (1-\Xi)^2|\gamma|^2\left\{\left[\frac{1+(1-\Xi)\,|\gamma|}{2(2+2\bar{\alpha})(8+24\bar{\alpha})}+\frac{1}{4(2+4\bar{\alpha})^2}\right]p_1^4\right. \\
&\qquad +\left[\frac{2+(1-\Xi)\,|\gamma|}{2(2+2\bar{\alpha})(8+24\bar{\alpha})}+\frac{2}{4(2+4\bar{\alpha})^2}\right]|x|\,p_1^2(4-p_1^2) \\
&\qquad +\frac{|x|^2p_1^2(4-p_1^2)}{2(2+2\bar{\alpha})(8+24\bar{\alpha})}+\frac{2p_1(4-p_1^2)}{2(2+2\bar{\alpha})(8+24\bar{\alpha})} \\
&\qquad \left.+\frac{|x|^2(16+p_1^4-8p_1^2)}{4(2+4\bar{\alpha})^2}\right\}.
\end{aligned}$$

Let the right side of inequality be $E(|x|,p_1)$, and differentiate E with respect to $|x|$,

$$\frac{\partial E(|x|,p_1)}{\partial |x|} = (1-\Xi)^2|\gamma|^2\left\{\left[\frac{2+(1-\Xi)\,|\gamma|}{2(2+2\bar{\alpha})(8+24\bar{\alpha})}+\frac{2}{4(2+4\bar{\alpha})^2}\right] p_1^2(4-p_1^2) + \frac{2\,|x|\,p_1^2(4-p_1^2)}{2(2+2\bar{\alpha})(8+24\bar{\alpha})}+\frac{2\,|x|\,(16+p_1^4-8p_1^2)}{4(2+4\bar{\alpha})^2}\right\} \geq 0.$$

It means that E is an increasing function of $|x|$. So maximum occurs when $|x|=1$, we obtain

$$\begin{aligned} &E(1,p_1)\\ &= (1-\Xi)^2|\gamma|^2\left\{\left[\frac{1+(1-\Xi)\,|\gamma|}{2(2+2\bar{\alpha})(8+24\bar{\alpha})}+\frac{1}{4(2+4\bar{\alpha})^2}\right]p_1^4 + \left[\frac{2+(1-\Xi)\,|\gamma|}{2(2+2\bar{\alpha})(8+24\bar{\alpha})}+\frac{2}{4(2+4\bar{\alpha})^2}\right]p_1^2(4-p_1^2) + \frac{p_1^2(4-p_1^2)}{2(2+2\bar{\alpha})(8+24\bar{\alpha})}+\frac{2p_1(4-p_1^2)}{2(2+2\bar{\alpha})(8+24\bar{\alpha})}+\frac{(16+p_1^4-8p_1^2)}{4(2+4\bar{\alpha})^2}\right\}\\ &= (1-\Xi)^2|\gamma|^2\left\{-\frac{1}{(2+2\bar{\alpha})(8+24\bar{\alpha})}p_1^4-\frac{1}{(2+2\bar{\alpha})(8+24\bar{\alpha})}p_1^3 + \frac{6+2(1-\Xi)\,|\gamma|}{(2+2\bar{\alpha})(8+24\bar{\alpha})}p_1^2+\frac{4}{(2+2\bar{\alpha})(8+24\bar{\alpha})}p_1+\frac{1}{(1+2\bar{\alpha})^2}\right\}\\ &= \frac{(1-\Xi)^2|\gamma|^2}{(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{-p_1^4-p_1^3+(6+2(1-\Xi)\,|\gamma|)p_1^2+4p_1 + \frac{(2+2\bar{\alpha})(8+24\bar{\alpha})}{(1+2\bar{\alpha})^2}\right\}. \end{aligned}$$

Differentiate $E(1,p_1)$ with respect to p_1,

$$\begin{aligned} &\frac{\partial E(1,p_1)}{\partial p_1}\\ &= \frac{(1-\Xi)^2|\gamma|^2}{(2+2\bar{\alpha})(8+24\bar{\alpha})}\{-4p_1^3-3p_1^2+(12+4(1-\Xi)\,|\gamma|)p_1+4\}\\ &= \frac{4(1-\Xi)^2|\gamma|^2}{(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{-p_1^3-\frac{3}{4}p_1^2+(3+(1-\Xi)\,|\gamma|)p_1+1\right\}. \end{aligned}$$

Let $\kappa = 3+(1-\Xi)\,|\gamma|$, $\mathfrak{A}(p_1) = -p_1^3-\frac{3}{4}p_1^2+\kappa p_1+1$. It is clear that $\kappa \geq 3$. When $\kappa = 3$, $\mathfrak{A}(p_1)$ has only one root in $[0,2]$, namely $p_1 \approx 1.568729$. Since $\mathfrak{A}$ is positive when $p_1 \in [0, 1.568729)$, so $E(1,p_1)$ is increasing in this domain. When $p_1 \in (1.568729, 2]$, $E(1,p_1)$ may be increasing or decreasing. Take λ be the value of p_1 which maximise E, which implies that

$$\begin{aligned} &\max E(1,p_1)\\ &= E(1,\lambda)\\ &= \frac{(1-\Xi)^2|\gamma|^2}{(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{-\lambda^4-\lambda^3+2\kappa\,\lambda^2+4\lambda+\frac{(2+2\bar{\alpha})(8+24\bar{\alpha})}{(1+2\bar{\alpha})^2}\right\} \end{aligned}$$

where λ must satisfy

$$-\lambda^3 - \frac{3}{4}\lambda^2 + \kappa\,\lambda + 1 = 0.$$

So we can further simplify the above expression as

$$\begin{aligned}
&\max E(1,p_1)\\
&= \frac{(1-\Xi)^2|\gamma|^2}{(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{\lambda\left(-\lambda^3 - \frac{3}{4}\lambda^2 + \kappa\lambda + 1\right) - \frac{1}{4}\lambda^3 + \kappa\lambda^2 + 3\lambda \right.\\
&\qquad\left. + \frac{(2+2\bar{\alpha})(8+24\bar{\alpha})}{(1+2\bar{\alpha})^2}\right\}\\
&= \frac{(1-\Xi)^2|\gamma|^2}{(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{-\frac{1}{4}\lambda^3 + \kappa\lambda^2 + 3\lambda + \frac{(2+2\bar{\alpha})(8+24\bar{\alpha})}{(1+2\bar{\alpha})^2}\right\}\\
&= \frac{(1-\Xi)^2|\gamma|^2}{(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{\frac{1}{4}\left(-\lambda^3 - \frac{3}{4}\lambda^2 + \kappa\lambda + 1\right) + \left(\kappa + \frac{3}{16}\right)\lambda^2 + \left(3 - \frac{\kappa}{4}\right)\lambda - \frac{1}{4}\right.\\
&\qquad\left. + \frac{(2+2\bar{\alpha})(8+24\bar{\alpha})}{(1+2\bar{\alpha})^2}\right\}\\
&= \frac{(1-\Xi)^2|\gamma|^2}{(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{\left(\frac{16\kappa+3}{16}\right)\lambda^2 + \left(\frac{12-\kappa}{4}\right)\lambda + \frac{188\bar{\alpha}^2 + 252\bar{\alpha} + 63}{4(1+2\bar{\alpha})^2}\right\}.
\end{aligned}$$

The quadratic function

$$y_1 = \left(\frac{16\kappa+3}{16}\right)\lambda^2 + \left(\frac{12-\kappa}{4}\right)\lambda + \frac{188\bar{\alpha}^2 + 252\bar{\alpha} + 63}{4(1+2\bar{\alpha})^2}$$

is maximum when $\lambda = \frac{2\kappa-24}{16\kappa+3}$, hence

$$\begin{aligned}
&\max E(1,p_1)\\
&= \frac{(1-\Xi)^2|\gamma|^2}{(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{\left(\frac{16\kappa+3}{16}\right)\left(\frac{2\kappa-24}{16\kappa+3}\right)^2 + \left(\frac{12-\kappa}{4}\right)\left(\frac{2\kappa-24}{16\kappa+3}\right)\right.\\
&\qquad\left. + \frac{188\bar{\alpha}^2 + 252\bar{\alpha} + 63}{4(1+2\bar{\alpha})^2}\right\}\\
&= \frac{(1-\Xi)^2|\gamma|^2}{(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{\frac{(\kappa-12)^2}{4(16\kappa+3)} - \frac{1}{2}\left(\frac{(\kappa-12)^2}{16\kappa+3}\right) + \frac{188\bar{\alpha}^2 + 252\bar{\alpha} + 63}{4(1+2\bar{\alpha})^2}\right\}\\
&= \frac{(1-\Xi)^2|\gamma|^2}{4(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{\frac{188\bar{\alpha}^2 + 252\bar{\alpha} + 63}{(1+2\bar{\alpha})^2} - \frac{(\kappa-12)^2}{16\kappa+3}\right\}.
\end{aligned}$$

For $\kappa \in [3,5]$, the expression is maximum when $\kappa = 5$, thus

$$\max E(1,p_1) = \frac{(1-\Xi)^2|\gamma|^2}{4(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{\frac{188\bar{\alpha}^2 + 252\bar{\alpha} + 63}{(1+2\bar{\alpha})^2} - \frac{49}{83}\right\}.$$

As κ increases, the positive root of $\mathfrak{A}(p_1)$ is also increases. When $\kappa > 5$, the positive root must exceed 2. In this scenario, $E(1,p_1)$ is always increasing for $p_1 \in [0,2]$. So we take $p_1 = 2$ (note that p_1 not necessarily maximise $\mathfrak{A}$),

$$
\begin{aligned}
&\max E(1,p_1)\\
&= E(1,2)\\
&= \frac{(1-\Xi)^2|\gamma|^2}{(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{-16-8+2\kappa\,(4)+8+\frac{(2+2\bar{\alpha})(8+24\bar{\alpha})}{(1+2\bar{\alpha})^2}\right\}\\
&= \frac{(1-\Xi)^2|\gamma|^2}{(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{8\kappa-16+\frac{1}{4}+\frac{188\bar{\alpha}^2+252\bar{\alpha}+63}{4(1+2\bar{\alpha})^2}\right\}\\
&= \frac{(1-\Xi)^2|\gamma|^2}{4(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{32\kappa-63+\frac{188\bar{\alpha}^2+252\bar{\alpha}+63}{(1+2\bar{\alpha})^2}\right\}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&|a_2a_4-a_3^2|\\
&\le \begin{cases}\dfrac{(1-\Xi)^2|\gamma|^2}{4(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{\dfrac{188\bar{\alpha}^2+252\bar{\alpha}+63}{(1+2\bar{\alpha})^2}-\dfrac{49}{83}\right\}, & \text{when } \kappa\in[3,5)\\ \dfrac{(1-\Xi)^2|\gamma|^2}{4(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{32\kappa-63+\dfrac{188\bar{\alpha}^2+252\bar{\alpha}+63}{(1+2\bar{\alpha})^2}\right\}, & \text{when } \kappa\ge 5\end{cases},
\end{aligned}
$$

where

$$\kappa = 3+(1-\Xi)\,|\gamma|. \quad \square$$

It is obvious that for all values of κ, we have

$$32\kappa-63+\frac{188\bar{\alpha}^2+252\bar{\alpha}+63}{(1+2\bar{\alpha})^2} > \frac{188\bar{\alpha}^2+252\bar{\alpha}+63}{(1+2\bar{\alpha})^2}-\frac{49}{83},$$

Hence

$$|a_2a_4-a_3^2| \le \frac{(1-\Xi)^2|\gamma|^2}{4(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{32\kappa-63+\frac{188\bar{\alpha}^2+252\bar{\alpha}+63}{(1+2\bar{\alpha})^2}\right\}.$$

For the functions defined by (2), equality can be attained in (9) by taking $p_1=2$ and $p_2=p_3=-2$.

The proof is now complete.

Theorem 3.7. *Suppose that f belongs to the class $\mathcal{S}^*_{s,\bar{\alpha},\gamma,\Xi}$, then the third Hankel determinant is given as*

$$
|H_3(1)| \le \begin{cases}\dfrac{|\gamma|^2}{216}\cdot\mathfrak{B}, & \textit{when } \rho<2\\ \dfrac{|\gamma|^2}{4}(5|\gamma|^2+12|\gamma|+3), & \textit{when } \rho\ge 2\end{cases},
$$

where

$$\mathfrak{B} = 9|\gamma|^2\sqrt{6|\gamma|+19} + 9|\gamma|\sqrt{6|\gamma|+19} + 207|\gamma|^2 + \left(557+19\sqrt{19}\right)|\gamma| + 134 + 19\sqrt{19}.$$

Proof. From Theorem 3.1, we have

$$|H_3(1)| \leq |a_3| \cdot \left|a_2a_4 - a_3^2\right| + |a_4| \cdot |a_4 - a_2a_3| + |a_5| \cdot \left|a_3 - a_2^2\right|$$

$$|H_3(1)| \leq \left[\frac{(1-\Xi)\,|\gamma|}{1+2\bar{\alpha}}\right] \cdot \left|a_2a_4 - a_3^2\right| + \left[\frac{(1-\Xi)|\gamma|\,[1+(1-\Xi)|\gamma|]}{2+6\bar{\alpha}}\right] \cdot |a_4 - a_2a_3| + \left[\frac{(1-\Xi)|\gamma|[1+(1-\Xi)|\gamma|]}{(2+8\bar{\alpha})}\right] \cdot \left|a_3 - a_2^2\right|.$$

Given that

$$\rho = -\frac{1}{3} + \sqrt{\frac{6(1-\Xi)|\gamma|(1+3\bar{\alpha}-2\bar{\alpha}^2) + 19(1+2\bar{\alpha})(1+\bar{\alpha})}{9(1+2\bar{\alpha})(1+\bar{\alpha})}},$$

and

$$\kappa = 3 + (1-\Xi)\,|\gamma|.$$

Applying Theorem 3.3, Theorem 3.5 and Theorem 3.6, we now encounter two cases,

(1) When $\rho < 2$, we have

$$|H_3(1)| \leq \left[\frac{(1-\Xi)\,|\gamma|}{1+2\bar{\alpha}}\right] \cdot \left[\frac{(1-\Xi)^2|\gamma|^2}{4(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{32\kappa - 63 + \frac{188\bar{\alpha}^2+252\bar{\alpha}+63}{(1+2\bar{\alpha})^2}\right\}\right] + \left[\frac{(1-\Xi)|\gamma|\,[1+(1-\Xi)|\gamma|]}{2+6\bar{\alpha}}\right] \cdot \left[\frac{(1-\Xi)|\gamma|}{8(1+3\bar{\alpha})}(-\rho^3-\rho^2+6\rho+4) + \frac{(1-\Xi)^2|\gamma|^2\,(1+3\bar{\alpha}-2\bar{\alpha}^2)\rho}{4(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}\right] + \left[\frac{(1-\Xi)|\gamma|[1+(1-\Xi)|\gamma|]}{(2+8\bar{\alpha})}\right] \cdot \left[\frac{(1-\Xi)^2\,|\gamma|^2}{(1+\bar{\alpha})^2} + \frac{(1-\Xi)\,|\gamma|}{1+2\bar{\alpha}}\right].$$

Expression on the right is maximum when $\Xi = 0 = \bar{\alpha}$. Under this setting, we have

$$\rho = -\frac{1}{3} + \sqrt{\frac{6|\gamma|+19}{9}} = \frac{\sqrt{6|\gamma|+19}-1}{3},$$

$$\kappa = 3 + |\gamma|$$

and $\max\limits_{\rho>0}(-\rho^3-\rho^2+6\rho+4) = \frac{52+38\sqrt{19}}{27}$, which is acquired when $\rho = \frac{-1+\sqrt{19}}{3}$. Hence

$$
\begin{aligned}
|H_3(1)| \leq |\gamma| \cdot \frac{32\,|\gamma|^3(3+|\gamma|)}{64} &+ \left[\frac{|\gamma|\,(1+|\gamma|)}{2}\right] \\
&\cdot \left[\frac{|\gamma|}{8}\left(\frac{52+38\sqrt{19}}{27}\right) + \frac{|\gamma|^2}{4}\left(\frac{\sqrt{6|\gamma|+19}-1}{3}\right)\right] + \left[\frac{|\gamma|(1+|\gamma|)}{2}\right] \\
&\cdot (|\gamma|^2 + |\gamma|)
\end{aligned}
$$

which can be simplified as

$$
\begin{aligned}
|H_3(1)| \leq \frac{|\gamma|^2}{216}\Big(9|\gamma|^2\sqrt{6|\gamma|+19} + 9|\gamma|\sqrt{6|\gamma|+19} + 207|\gamma|^2 \\
+ \left(557+19\sqrt{19}\right)|\gamma| + 134 + 19\sqrt{19}\Big).
\end{aligned}
$$

(2) When $\rho \geq 2$, we have

$$
\begin{aligned}
|H_3(1)| \leq \left[\frac{(1-\Xi)\,|\gamma|}{1+2\bar{\alpha}}\right] \cdot \left[\frac{(1-\Xi)^2|\gamma|^2}{4(2+2\bar{\alpha})(8+24\bar{\alpha})}\left\{32\kappa - 63 + \frac{188\bar{\alpha}^2+252\bar{\alpha}+63}{(1+2\bar{\alpha})^2}\right\}\right] \\
+ \left[\frac{(1-\Xi)|\gamma|\,[1+(1-\Xi)|\gamma|]}{2+6\bar{\alpha}}\right] \\
\cdot \left[\frac{(1-\Xi)|\gamma|}{2(1+3\bar{\alpha})} + \frac{(1-\Xi)^2|\gamma|^2\,(1+3\bar{\alpha}-2\bar{\alpha}^2)}{2(1+\bar{\alpha})(1+2\bar{\alpha})(1+3\bar{\alpha})}\right] \\
+ \left[\frac{(1-\Xi)|\gamma|[1+(1-\Xi)|\gamma|]}{(2+8\bar{\alpha})}\right] \cdot \left[\frac{(1-\Xi)^2\,|\gamma|^2}{(1+\bar{\alpha})^2} + \frac{(1-\Xi)\,|\gamma|}{1+2\bar{\alpha}}\right].
\end{aligned}
$$

Following the arrangements similar with case (1), we obtain

$$
|H_3(1)| \leq \frac{|\gamma|^2}{4}(5|\gamma|^2 + 12|\gamma| + 3).
$$

Equality is attainable by taking all upper bounds from Theorem 3.1, 3.3, 3.5 and 3.6.

This end the proof. □

4. Conclusion

In this article, a subclass of univalent function associated with generalised Sakaguchi function is defined. Several coefficient bounds are found and some inequalities of certain determinants are evaluated. Finally, the third Hankel determinant is assessed. For Theorem 3.5, we are unable to identify which among the two expressions would provide the larger value, so we consider both cases.

Acknowledgments

The authors would like to thank UKM for supporting the work written in this article. We would also like to thanks the referees for constructive comments.

References

Babalola K.O. 2007. On $H_3(1)$ Hankel determinant for some classes of univalent functions. *International Journal of Inequalities and Application* **6**: 1–7.

Bansal D., Maharana S. & Prajapat J.K. 2015. Third order Hankel determinant for certain univalent functions. *Journal of the Korean Mathematical Society.* **52**(6): 1139–1148.

Das R.N. & Singh P. 1977. On subclasses of schlicht mapping. *Indian Journal of Pure and Applied Mathematics* **8**: 864–872.

Duren P.L. 1983. *Univalent Functions*. New York: Springer-Verlag.

Fekete M. & Szegö G. 1933. Eine bemerkung über ungerade schlichte funktionen. *Journal of the London Mathematical Society* **s1-8**(2): 85–89.

Hayman W.K. 1968. On second Hankel determinant of mean univalent functions. *Proceedings of the London Mathematical Society* **s3-18**(1): 77–94.

Janteng A., Halim S.A. & Darus M. 2006. Coefficient inequality for a function whose derivative has a positive real part. *Journal of Inequalities in Pure and Applied Mathematics* **7**(2): 50.

Keogh F.R. & Merkes E.P. 1969. A coefficient inequality for certain classes of analytic functions. *Proceedings of the American Mathematical Society* **20**(1): 8–12.

Libera R.J. & Złotkiewicz E.J. 1982. Early coefficients of the inverse of a regular convex function. *Proceedings of the American Mathematical Society* **85**(2): 225–230.

Libera R.J. & Złotkiewicz E.J. 1983. Coefficients bounds for the inverse of a function with derivative in $\mathcal{P}$. *Proceedings of the American Mathematical Society* **87**(2): 251–257.

Mishra A.K., Prajapat J.K. & Maharana S. 2016. Bounds on Hankel determinant for starlike and convex functions with respect to symmetric points. *Cogent Mathematics* **3**(1): 1160557.

Noonan J.W. & Thomas D.K. 1976. On the second Hankel determinant of areally mean p-valent functions. *Transactions of the American Mathematical Society* **223**(2): 337–346.

Pommerenke C. 1966. On the coefficients and Hankel determinant of univalent functions. *Journal of the London Mathematical Society* **s1-41**(1): 111–112.

Raza M. & Malik S.N. 2013. Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli. *Journal of Inequality and Applications* **2013**(1): 412.

Sakaguchi K. 1959. On a certain univalent mapping. *Journal of the Mathematical Society of Japan* **11**(1): 72–75.

Selvaraj C. & Vasanthi N. 2011. Subclasses of analytic functions with respect to symmetric and conjugate points. *Tamkang Journal of Mathematics* **42**(1): 87–94.

Singh G., Singh G. & Singh G. 2019. Certain subclasses of Sakaguchi type bi-univalent functions. *GANITA* **69**(2): 45–55.

Vanitha L., Dhanalakshmi K. & Ramachandran C. 2018. On third order Hankel determinant for some special claas of analytic functions related with generalized Sakaguchi functions. *International Journal of Mathematics and Its Applications* **6**(4): 163–173.

Department of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 UKM Bangi
Selangor, MALAYSIA
E-mail: p154492@siswa.ukm.edu.my, maslina@ukm.edu.my*

Received: 11 June 2025
Accepted: 30 January 2026

*Corresponding author