

PARAMETRIZED TRAPEZOIDAL NEUTROSOPHIC SET AND ITS APPLICATION TO MULTI-ATTRIBUTE DECISION-MAKING

(Set Neutrosoph Trapezoid Berparameter dan Aplikasinya kepada Pembuatan Keputusan Pelbagai Atribut)

MOHD AZDI MAASAR, ZAHARI MD RODZI*, ZAIFILLA FARRINA ZAINUDIN,
MOHAMMAD SYAFIQ MOHAMMAD KAMARI & FAISAL AL-SHARQI

ABSTRACT

In modern engineering, management, and information systems, decision-making is faced with a variety of conflicting traits, various heterogeneous information sources, and pervasive uncertainty. Typical multi-attribute decision-making (MADM) techniques are not suitable for modeling the indeterminacy, nonlinear preference behavior, and asymmetric risk attitude present in real-world decision-making scenarios. To overcome these limitations, this paper introduces a new concept of the Parametrized Trapezoidal Neutrosophic Set (PTNS) to fully extend Ye's traditional trapezoidal neutrosophic numbers concept by adding truth- and falsity-based influence regulation parameters (α, β) for nonlinear shapes. On this basis, rank and accuracy functions are designed in parametric form and derived from the proposed PTNS. This keeps the full compatibility with traditional neutrosophic ranking functions in the linear case (where $(\alpha, \beta) = (1, 1)$). In addition, a decision-making framework for PTNS-SYAI is proposed by combining the parametric scoring measure with the Simplified Yielded Aggregation Index (SYAI), enabling efficient normalization, aggregation, and ranking of alternatives under uncertainty. A well-defined, easily understood software selection problem from the literature is presented to illustrate the proposed approach's ability and interpretability. Key properties of the proposed score, such as boundedness, monotonicity, smoothness in (α, β) , and exact reduction to the classical model at $(\alpha, \beta) = (1, 1)$, are formally demonstrated. Under varying (α, β) setups, sensitivity analysis reveals that the system gracefully rebalances the ranks, solving the real closeness of ties with justifiable results, but preserving the rank order of alternatives when no significant closeness exists. The ranking results of the proposed method are compared with those of TOPSIS, VIKOR, and COPRAS in a cross-method comparison of the second-largest problem, demonstrating consistent rankings. The outcomes established that the PTNS-SYAI framework is a powerful, flexible, and low-cost decision-making tool for complex, multi-attribute, and uncertain decision problems.

Keywords: parametrized neutrosophic set; trapezoidal neutrosophic numbers; multi-attribute decision-making; SYAI method

ABSTRAK

Masalah pembuatan keputusan dalam bidang kejuruteraan, pengurusan, dan sistem maklumat kontemporari semakin dicirikan oleh pelbagai atribut yang bercanggah, sumber maklumat yang heterogen, dan ketidakpastian yang meluas. Kaedah pembuatan keputusan pelbagai atribut (MADM) klasik dan model kabur linear sering gagal menangkap ketidaktentuan, tingkah laku keutamaan tak linear, dan sikap risiko tidak simetri yang wujud dalam persekitaran keputusan dunia nyata dengan secukupnya. Bagi mengatasi batasan ini, makalah ini mencadangkan Set Neutrosoph Trapezoid Berparameter (PTNS) yang baharu, yang memperluaskan nombor neutrosoph trapezoid klasik Ye dengan menggabungkan parameter bentuk tak linear (α, β) untuk mengawal pengaruh komponen kebenaran dan kepalsuan secara geometri. Berdasarkan struktur PTNS yang dicadangkan, fungsi skor dan ketepatan berparameter dibangunkan, yang mengitlakkan mekanisme pemeringkatan neutrosoph tradisional sambil mengekalkan keserasian ke belakang sepenuhnya dengan kes linear apabila $(\alpha, \beta) = (1, 1)$. Tambahan pula, rangka kerja

keputusan PTNS–SYAI bersepadu dibina dengan menggandingkan skim pemarkahan berparameter dengan Indeks Pengagregatan Terhasil Ringkas (SYAI) untuk menormal, mengagregat, dan memeringkat alternatif di bawah ketidakpastian. Keberkesanan dan kebolehtafsiran pendekatan ini dibuktikan melalui aplikasi pemilihan perisian ilustratif yang dibentangkan. Sifat-sifat utama skor yang dicadangkan, termasuk keterbatasan, kemonotonan data, kelicinan terhadap (α, β) , dan penurunan tepat kepada model klasik pada $(\alpha, \beta) = (1, 1)$, dibuktikan secara formal. Analisis sensitiviti menunjukkan rangka kerja ini melaraskan pemeringkatan secara terkawal bagi menyelesaikan keadaan hampir-seri sambil mengekalkan alternatif yang jelas terpisah. Perbandingan merentas kaedah pada masalah kedua yang lebih besar menunjukkan bahawa pemeringkatan yang terhasil selaras dengan kaedah TOPSIS, VIKOR, dan COPRAS, sekali gus mengesahkan peringkat pengagregatan. Hasil kajian membuktikan rangka kerja PTNS–SYAI menawarkan alat yang teguh, fleksibel, dan cekap dari segi pengiraan untuk analitis keputusan pelbagai atribut.

Kata kunci: set neutrosoph berparameter; nombor neutrosoph trapezoid; pembuatan keputusan pelbagai atribut; kaedah SYAI

1. Introduction

Decision-making problems in modern engineering, management, and information systems are becoming increasingly complex, involving multidimensional criteria, heterogeneous information sources, and pervasive uncertainty. Under these conditions, decision-makers must consider multiple alternatives while balancing conflicting benefit and cost criteria, subjective expert opinions, and the presence of partial or ill-defined information. Such problems are difficult to address using classical optimisation and crisp multiple-criteria decision-making tools, thereby motivating the development of new mathematical models for uncertainty-driven decision analysis.

Multi-attribute decision-making (MADM) provides a fundamental paradigm for addressing such problems by organising the evaluation and ranking of alternatives across multiple attributes. Traditional MADM methods, such as the Analytic Hierarchy Process (AHP) and TOPSIS, as well as aggregation-based approaches, provide systematic mechanisms for preference modelling and the comparison of alternatives. However, these methods generally assume that input information is precise and numerical and that aggregation is linear. Such assumptions are often not satisfied in real-world decision contexts, where expert judgements may be vague, hesitant, and sometimes conflicting.

These limitations have motivated the use of fuzzy set theory and its extensions in many MADM frameworks to represent imprecision and uncertainty. Although fuzzy and intuitionistic fuzzy models provide greater expressive capability, they do not explicitly represent indeterminacy, which plays an important role in human judgement in complex decision-making environments. To address this limitation, neutrosophic set theory provides separate representations of the degrees of truth, indeterminacy, and falsity.

In neutrosophic models, trapezoidal neutrosophic numbers (TNNs) proposed by Ye (2015) have attracted considerable attention because of their computational simplicity and applicability to multi-criteria evaluation problems. TNNs provide decision-makers with an interpretable and structured means of representing uncertain information through trapezoidal structures associated with the neutrosophic components. Score and accuracy functions can then be used to transform neutrosophic evaluations into scalar values for ranking alternatives.

Despite these advantages, most existing TNN-based MADM methods employ linear membership functions and fixed trapezoidal shapes, which may not adequately capture decision-makers' preferences and asymmetric attitudes towards uncertainty. Recent studies have increasingly recognised that decision contexts may involve nonlinear relationships among criteria

and differing attitudes towards risk, optimism, and conservatism. Consequently, linear neutrosophic formulations may oversimplify the evaluation process and restrict both the flexibility and interpretability of the resulting models. This motivates the development of parametric and non-linear extensions of trapezoidal neutrosophic representations, in which the effects of the truth and falsity components can be adaptively controlled to represent a wider range of behavioural preferences.

At the same time, aggregation and ranking mechanisms play an important role in transforming neutrosophic evaluations into actionable decisions. Similarity-based approaches, score-driven ranking methods, and TOPSIS-like techniques have been widely studied in neutrosophic MADM. In this context, the Simplified Yielded Aggregation Index (SYAI) has recently been proposed as an efficient and interpretable aggregation framework that is simple to implement while retaining strong discriminative capability (Rahman *et al.* 2025). However, most current applications of SYAI and related approaches make restrictive assumptions regarding the shape of uncertainty representations and generally apply the same structural form to all judgements.

This paper introduces a new *Parametrized Trapezoidal Neutrosophic Set* (PTNS) framework for MADM, in which trapezoidal neutrosophic numbers are extended using nonlinear shape parameters (α, β) . These parameters provide explicit control over the curvature and influence of the truth and falsity components, thereby allowing interpretable adjustments within a unified mathematical structure. The PTNS framework is subsequently integrated with the SYAI method to develop a consistent PTNS–SYAI decision-making approach that facilitates efficient aggregation and ranking under uncertainty.

The main contributions of this paper are summarised as follows:

- A parametrized trapezoidal neutrosophic set (PTNS) is introduced by extending Ye’s classical trapezoidal neutrosophic numbers with two shape parameters, (α, β) , which control the curvature of the ascending and descending membership segments.
- A parametric score and accuracy pair is derived from the centroid of the parametrized membership function. Consequently, the shape parameters act through the membership geometry rather than through external exponents, while truth, indeterminacy, and falsity are treated using a common operator.
- The proposed score is shown to be bounded, monotone with respect to the data, smooth and monotone in (α, β) , well behaved under extreme parameter values, and exactly reducible to the classical model when $(\alpha, \beta) = (1, 1)$.
- An integrated PTNS–SYAI MADM framework is developed to provide a transparent and computationally efficient procedure for evaluating and ranking alternatives.
- Two illustrative examples are presented. The first is a software selection problem involving quantitative sensitivity and stability analyses, while the second is a larger provider selection problem in which the resulting rankings are cross-validated using TOPSIS, VIKOR, and COPRAS.

The remainder of this paper is organised as follows. Section 2 reviews previous studies on MADM, neutrosophic sets, parametric modelling, and neutrosophic-based ranking methods. Section 3 introduces the proposed parametrized trapezoidal neutrosophic set, together with its operations and ranking functions. Section 4 presents a representative software selection example to illustrate the implementation of the PTNS–SYAI framework, followed by sensitivity and comparative analyses. Finally, Section 5 concludes the paper and outlines directions for future research.

2. Literature Review

2.1. Multi-Attribute Decision-Making (MADM): Concepts and Challenges

Multi-attribute decision-making (MADM) has developed considerably to address the multi-dimensional nature of attributes and uncertainty in decision-making problems. In principle,

MADM enables the evaluation and selection of alternatives based on multiple criteria, including both qualitative and quantitative attributes (Martinkutė-Kaulienė *et al.* 2021). This decision-making framework is designed to assist decision-makers in navigating complex evaluation environments while formally representing stakeholders' preferences and trade-offs.

An important aspect of MADM is the distinction between benefit and cost criteria, which enables a balanced quantitative evaluation of candidate alternatives. The relative importance assigned to attributes may vary considerably according to the preferences of decision-makers and the nature of the criteria involved. Multi-criteria decision-making (MCDM) methods, including the Analytic Hierarchy Process (AHP) and TOPSIS, have been widely applied because they provide systematic and intuitive frameworks for evaluating alternatives (Fu *et al.* 2020). These approaches determine rankings by combining assessments across multiple dimensions.

However, traditional MADM approaches encounter significant difficulties when expert judgements are vague or imprecise. Human evaluators often express uncertainty with varying levels of confidence that cannot be adequately captured by crisp or strictly linear models (Li & Yang 2024). These difficulties become more pronounced in decision problems involving nonlinear relationships among attributes, thereby creating a need for more sophisticated modelling techniques.

In such situations, fuzzy set theory remains one of the most widely used approaches for representing imprecision and uncertainty (Khan *et al.* 2023). Extensions such as intuitionistic fuzzy sets and hesitant fuzzy sets provide more detailed representations of expert judgements and uncertainty in attribute weights (Alghazzawi *et al.* 2023). Furthermore, complex aggregation and inference mechanisms may be required to accommodate nonlinear relationships and heterogeneous uncertainty structures. Recent studies involving probabilistic fuzzy modelling and alternative aggregation paradigms have demonstrated improved robustness and reliability in complex MADM problems (Li *et al.* 2020). These developments highlight the continuing need for decision-making models capable of addressing the increasing complexity of contemporary multi-attribute environments.

2.2. *Neutrosophic Sets and Trapezoidal Neutrosophic Numbers in Decision-Making*

Neutrosophic sets (NSs), introduced by Smarandache, provide a mathematical framework for handling uncertainty and indeterminacy in decision-making processes through three distinct membership components: truth, indeterminacy, and falsity. Unlike classical fuzzy and intuitionistic fuzzy sets, neutrosophic sets explicitly represent indeterminate information, thereby enhancing their ability to capture ambiguity, inconsistency, and incomplete information in real-world decision environments (Martinkutė-Kaulienė *et al.* 2021).

Neutrosophic sets have subsequently been applied in various fields, including engineering evaluation, risk analysis, and management science, particularly when decision information is vague or contradictory (Zhang *et al.* 2023). To facilitate their application in multi-criteria decision-making, Ye (2015) introduced trapezoidal neutrosophic numbers (TNNs), which provide a computationally manageable representation of neutrosophic information. In this framework, the truth, indeterminacy, and falsity components are represented using trapezoidal structures, thereby allowing vagueness and uncertainty in subjective evaluations to be modelled more effectively (Li & Yang 2024).

This representation enables decision-makers to express evaluation information in a more flexible and realistic manner, particularly in MCDM environments in which expert judgement plays a central role (Khan *et al.* 2023; Alghazzawi *et al.* 2023). In trapezoidal neutrosophic decision frameworks, score and accuracy functions serve as important mechanisms for ranking and comparing alternatives. Score functions combine information relating to truth, indeterminacy, and falsity into a scalar measure, while accuracy functions provide additional discriminatory information when alternatives have similar score values (Delcea *et al.* 2023).

Empirical analyses have shown that the combined use of score and accuracy measures can improve discrimination among alternatives, particularly when decision information is incom-

plete or conflicting (Reig-Mullor & Salas-Molina 2022). Despite these methodological advances, linear neutrosophic modelling approaches have several limitations. Linear formulations may oversimplify complex decision-making situations in which interactions among attributes are nonlinear and the effects of uncertainty are asymmetric (Kalaivani & Kaliyaperumal 2023).

These limitations are particularly relevant in areas such as environmental assessment, resource management, and technological evaluation, where decision variables may interact through complex and non-additive relationships. Recent studies have highlighted the need for modelling approaches capable of representing such relationships more accurately than conventional linear neutrosophic models (Gonul Bilgin *et al.* 2022). Although neutrosophic sets and trapezoidal neutrosophic numbers represent an important advancement in uncertainty modelling for decision-making, the limitations of linear neutrosophic formulations motivate the development of more flexible frameworks capable of representing nonlinear and adaptive uncertainty. This issue forms the basis for the parametric extension proposed in this study.

2.3. Parametric and Nonlinear Extensions of Neutrosophic Modelling

Trapezoidal neutrosophic numbers provide an effective means of representing uncertainty; however, the existing literature has identified several limitations associated with their linear structure and fixed-shape representation. Some studies have therefore proposed nonlinear and parametric extensions of neutrosophic models to provide greater flexibility, particularly in dynamic and complex decision environments.

For instance, simplified nonlinear neutrosophic sets have been proposed to better represent dynamic decision contexts (Qiu *et al.* 2024), while other studies have integrated neutrosophic representations with hybrid evaluation structures in emerging application areas (Mohamed *et al.* 2024). The integration of neutrosophic representations with DEMATEL–ANP and advanced aggregation operators further highlights the need for more sophisticated and adaptive representations of uncertainty in related decision-making frameworks (Çelikkilek 2025; Thong *et al.* 2025).

The major contributions of previous studies and their corresponding limitations are synthesised in Table 1 to clarify the motivation for the proposed parametrized trapezoidal neutrosophic set (PTNS) with tunable shape parameters (α, β) .

Recent developments in the literature also indicate a need for extensions of neutrosophic sets in which the influence and shape of the neutrosophic components can be explicitly controlled. Introducing the parameters (α, β) enables nonlinear modulation of the truth and falsity components, thereby making the framework more adaptable and interpretable in terms of its behaviour. This consideration motivates the PTNS framework proposed in this study.

Rather than applying the parameters as external powers to the membership values, the proposed framework incorporates the shape parameters into a derived centroid-based score, as discussed in Section 3.4. The same formulation is applied consistently to the neutrosophic components and reduces exactly to the classical model when $(\alpha, \beta) = (1, 1)$. This approach differs from q-rung and spherical neutrosophic generalisations (Khan *et al.* 2023; Abdullah *et al.* 2024), which extend the admissible range of membership degrees rather than the shape of the membership function, and from fixed-shape nonlinear models that do not provide an exact backward-compatible reduction.

2.4. MADM Methods: Ranking, Similarity, and Aggregation Models

The development of neutrosophic-based multi-attribute decision-making (MADM) methods has been driven by the need to address complex decision problems involving uncertainty, indeterminacy, and nonlinear interactions among attributes. Existing methods generally focus on ranking, similarity measurement, and aggregation mechanisms, employing various mathematical tools to improve the accuracy, robustness, and interpretability of decision outcomes. This subsection reviews several key methodological categories and positions the proposed integration

of parametrized trapezoidal neutrosophic sets (PTNS) with the Simplified Yielded Aggregation Index (SYAI) within the evolving research landscape.

Table 1: Representative studies on nonlinear and parametric neutrosophic modelling and motivation for PTNS

Reference	Summary of contributions	Gaps and limitations	Motivation for PTNS
Qiu <i>et al.</i> (2024)	Proposes dynamic nonlinear simplified neutrosophic sets for group decision-making in evolving environments.	Nonlinear behaviour is not explicitly shape-controlled, and trapezoidal structures are not parametrized.	PTNS introduces explicit shape parameters (α, β) to control nonlinear truth and falsity effects.
Mohamed <i>et al.</i> (2024)	Develops a hybrid neutrosophic evaluation framework for assessing Industry 5.0 enablers.	Fixed trapezoidal neutrosophic representations restrict modelling flexibility and behavioural expressiveness.	PTNS enhances uncertainty representation through tunable trapezoidal shapes.
Çelikkilek (2025)	Applies an integrated neutrosophic DEMATEL–ANP approach to renewable energy source evaluation.	Linear aggregation fails to capture complex interdependencies among decision criteria.	PTNS allows nonlinear modulation of neutrosophic components to better reflect complex interactions.
Thong <i>et al.</i> (2025)	Introduces enhanced decision-making models using advanced aggregation operators in generalized neutrosophic environments.	Existing approaches rely on fixed membership structures without explicit shape variability.	PTNS provides a tractable parametric extension that enables flexible shape adjustment in trapezoidal neutrosophic numbers.
Khan <i>et al.</i> (2023)	Develops q-rung orthopair fuzzy hypersoft aggregation operators for supplier selection.	Raises membership and non-membership degrees to a power q at the set level; indeterminacy is not modelled independently, and the trapezoidal shape is not parametrized.	PTNS retains the full neutrosophic triple and parametrizes the membership geometry rather than using an orthopair exponent.
Abdullah <i>et al.</i> (2024); Eroğlu (2025)	Apply spherical and interval-valued neutrosophic models within TOPSIS/VIKOR ranking frameworks.	Broaden the domain of the membership degrees but retain fixed membership structures and external ranking mechanisms.	PTNS instead generalizes the membership shape through (α, β) and derives the score directly from that geometry.

2.4.1. Simplified yielded aggregation index (SYAI)

The Simplified Yielded Aggregation Index (SYAI) provides a relatively straightforward mechanism for aggregating criterion-level information and ranking alternatives through a unified index. Its computational structure is particularly suitable for the proposed PTNS framework because the parametrized score generated by PTNS can be incorporated directly into the subsequent normalisation and aggregation stages.

SYAI is adopted in this study instead of TOPSIS, VIKOR, COPRAS, or TODIM because its structure is compatible with the proposed PTNS score and avoids the additional ideal- and

anti-ideal-distance calculations required by TOPSIS and the compromise parameter used in VIKOR. This preserves the transparency of the decision-making pipeline after the parametrized score has been calculated. Furthermore, the normalisation stage in SYAI operates smoothly on the resulting crisp scores, allowing controlled variations introduced by (α, β) at the scoring stage to remain visible in the final rankings.

As demonstrated in Section 4.4, the rankings produced using the proposed PTNS–SYAI framework are subsequently compared with those obtained using TOPSIS, VIKOR, and COPRAS on an independent decision problem. This comparative analysis is used to assess the consistency and discriminatory capability of the proposed framework while demonstrating that the parametrized score can also be incorporated into other established MADM procedures.

2.4.2. Similarity-based models

Neutrosophic MADM models based on similarity measures evaluate alternatives according to their closeness to ideal or reference solutions under uncertain information. Many of these methods employ advanced mathematical operators, such as Dombi or other generalised aggregation functions, to represent nonlinear similarity relationships. Empirical studies have shown that similarity-based models can improve the sensitivity and discriminatory capability of rankings, particularly when decision information exhibits substantial vagueness and indeterminacy (Awajan *et al.* 2021; Radha *et al.* 2025). These methods are useful for comparative evaluation tasks, although their performance may depend on appropriate calibration to maintain ranking stability across different decision contexts.

2.4.3. Score-based ranking techniques

Score-based ranking techniques remain widely used in neutrosophic MADM because of their conceptual simplicity and computational efficiency. These methods transform neutrosophic evaluations into scalar values using score and accuracy functions, thereby allowing alternatives to be compared and ranked directly. More recently, refined score functions have been proposed to improve discrimination among alternatives under complex uncertainty structures (Elsayed & Mohamed 2024). Despite their effectiveness, conventional score-based approaches often rely on linear combinations of neutrosophic components and may therefore be unable to represent nonlinear decision-maker preferences adequately.

2.4.4. TOPSIS-like neutrosophic methods

TOPSIS and its neutrosophic extensions remain widely used because of their intuitive principle of ranking alternatives according to their relative distances from ideal and anti-ideal solutions. Several adaptations have been developed for neutrosophic information, including interval-valued neutrosophic TOPSIS (Abdullah *et al.* 2024) and spherical neutrosophic TOPSIS (Eroğlu 2025). These methods provide mathematically structured and flexible decision-making procedures, particularly in engineering and management applications. However, their effectiveness depends on the underlying neutrosophic representation and distance measures, which may not always capture nonlinear preference behaviour or complex interactions among decision attributes (Mahmood & Ali 2022).

2.4.5. Positioning the PTNS–SYAI integration

Although numerous neutrosophic-based MADM methods have been developed, many continue to employ fixed-shape neutrosophic representations that provide limited control over nonlinear preference behaviour and varying attitudes towards uncertainty. The proposed PTNS–SYAI framework addresses this limitation by introducing adjustable shape parameters (α, β) at the

neutrosophic modelling stage and subsequently integrating the resulting parametrized scores into the SYAI ranking procedure.

The PTNS component enables nonlinear modulation of the truth and falsity components through the geometry of the membership functions, while SYAI provides a transparent and computationally efficient mechanism for aggregating and ranking the resulting evaluations. Recent studies have emphasised the importance of hybrid and adaptive decision-making frameworks for improving flexibility and robustness under uncertain environments (Rahman *et al.* 2023; Li *et al.* 2023; Mythili *et al.* 2025).

Overall, neutrosophic-based MADM methods encompass a broad range of ranking, similarity, and aggregation approaches that provide valuable mechanisms for decision-making under uncertainty. The integration of PTNS and SYAI provides a unified framework that combines flexible uncertainty representation, nonlinear expressiveness, and computational efficiency, thereby establishing a foundation for more sophisticated decision analysis in complex and uncertain environments.

3. Parametrized Trapezoidal Neutrosophic Set

This section generalises the classical trapezoidal neutrosophic set introduced by Ye (2015) by incorporating nonlinear shape parameters into the truth, indeterminacy, and falsity membership structures. In the original model, the membership functions are piecewise linear and are determined by the four vertices of a trapezoid. Although this structure is computationally simple, it may not adequately represent situations in which membership values increase or decrease at nonlinear rates, exhibit asymmetric behaviour, or respond differently to contextual conditions.

To address this limitation, we introduce a *parametrized trapezoidal neutrosophic set* in which each trapezoidal component is augmented with two shape-controlling parameters, (α, β) , governing the curvature of the ascending and descending segments. When $(\alpha, \beta) = (1, 1)$, the proposed model reduces exactly to the conventional trapezoidal neutrosophic set, thereby preserving compatibility with the foundational definitions in Ye (2015). The proposed parametrization provides a more flexible and expressive structure for modelling nonlinear attitudes towards truth, indeterminacy, and falsity.

3.1. Parametrized Trapezoidal Membership Function

Let (z_1, z_2, z_3, z_4) be a quadruple of real numbers satisfying

$$0 \leq z_1 < z_2 \leq z_3 < z_4 \leq 1.$$

In Ye's formulation, the corresponding trapezoidal membership function is linear on $[z_1, z_2]$ and $[z_3, z_4]$. We generalise this structure by introducing nonlinear shape parameters.

Definition 3.1 (Parametrized trapezoidal membership function). For $\alpha > 0$ and $\beta > 0$, the (α, β) -parametrized trapezoidal membership function associated with the quadruple (z_1, z_2, z_3, z_4) is defined by

$$\mu_{[z_1, z_2, z_3, z_4]}^{(\alpha, \beta)}(x) = \begin{cases} 0, & x \leq z_1, \\ \left(\frac{x - z_1}{z_2 - z_1} \right)^\alpha, & z_1 < x \leq z_2, \\ 1, & z_2 < x < z_3, \\ \left(\frac{z_4 - x}{z_4 - z_3} \right)^\beta, & z_3 \leq x < z_4, \\ 0, & x \geq z_4. \end{cases} \quad (1)$$

Definition 3.1 replaces the linear slopes in Ye's trapezoidal membership function with curvature-controlled nonlinear transitions. The parameter α governs the rate of increase on $[z_1, z_2]$, while β controls the rate of decrease on $[z_3, z_4]$. When $(\alpha, \beta) = (1, 1)$, the function reduces to the classical trapezoidal form, thereby ensuring backward compatibility with the conventional model.

Example 3.2 (Numerical illustration of Definition 3.1). Consider the quadruple

$$(z_1, z_2, z_3, z_4) = (0.2, 0.4, 0.7, 0.9)$$

with shape parameters $\alpha = 2$ and $\beta = 0.5$. According to Eq. 1, the corresponding (α, β) -parametrized trapezoidal membership function is

$$\mu_{[0.2, 0.4, 0.7, 0.9]}^{(2, 0.5)}(x) = \begin{cases} 0, & x \leq 0.2, \\ \left(\frac{x - 0.2}{0.4 - 0.2}\right)^2, & 0.2 < x \leq 0.4, \\ 1, & 0.4 < x < 0.7, \\ \left(\frac{0.9 - x}{0.9 - 0.7}\right)^{0.5}, & 0.7 \leq x < 0.9, \\ 0, & x \geq 0.9. \end{cases}$$

The membership degree is evaluated at three representative points.

(i) *Point in the ascending region*, $x = 0.3$. Since $0.2 < 0.3 \leq 0.4$,

$$\mu_{[0.2, 0.4, 0.7, 0.9]}^{(2, 0.5)}(0.3) = \left(\frac{0.3 - 0.2}{0.4 - 0.2}\right)^2 = \left(\frac{0.1}{0.2}\right)^2 = (0.5)^2 = 0.25.$$

(ii) *Point in the core (plateau) region*, $x = 0.5$. Since $0.4 < 0.5 < 0.7$, Definition 3.1 gives

$$\mu_{[0.2, 0.4, 0.7, 0.9]}^{(2, 0.5)}(0.5) = 1.$$

(iii) *Point in the descending region*, $x = 0.8$. Since $0.7 \leq 0.8 < 0.9$,

$$\mu_{[0.2, 0.4, 0.7, 0.9]}^{(2, 0.5)}(0.8) = \left(\frac{0.9 - 0.8}{0.9 - 0.7}\right)^{0.5} = \left(\frac{0.1}{0.2}\right)^{0.5} = (0.5)^{0.5} \approx 0.7071.$$

This example illustrates how the proposed shape parameters (α, β) govern the nonlinearity of the ascending and descending segments: the choice $\alpha = 2$ makes the rise from 0 to 1 more gradual, while $\beta = 0.5$ makes the decay from 1 to 0 slower and more curved compared to the linear case $(\alpha, \beta) = (1, 1)$.

3.2. Parametrized Trapezoidal Neutrosophic Number

Following Ye (2015), a trapezoidal neutrosophic number consists of three trapezoidal components representing truth, indeterminacy, and falsity. We retain this triadic structure and denote the breakpoints of the three components by (t_1, t_2, t_3, t_4) , (u_1, u_2, u_3, u_4) , and (v_1, v_2, v_3, v_4) , respectively, to distinguish their roles clearly within the proposed parametrized framework.

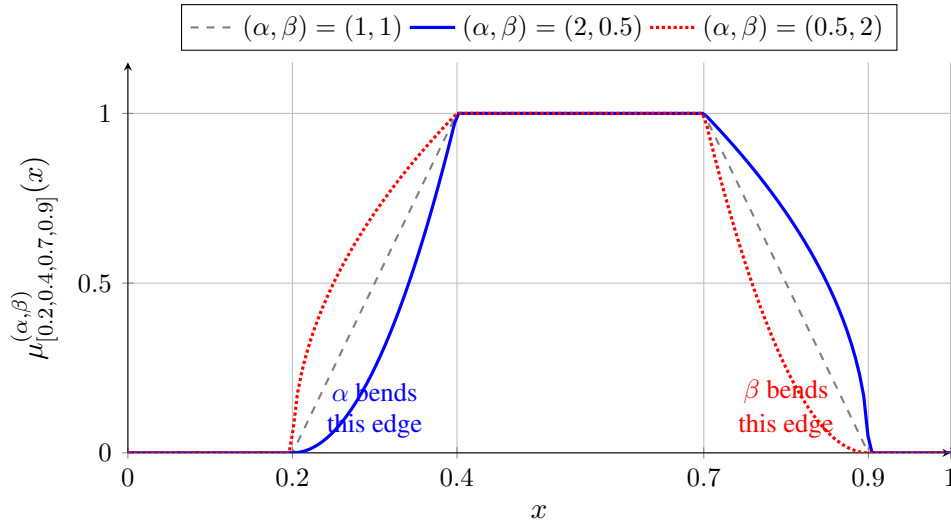


Figure 1: The parametrized trapezoidal membership function of Definition 3.1 on the quadruple $(0.2, 0.4, 0.7, 0.9)$. The dashed grey curve is the linear case $(\alpha, \beta) = (1, 1)$. The parameter α controls the curvature of the ascending segment $[z_1, z_2]$ and β that of the descending segment $[z_3, z_4]$: $\alpha > 1$ delays the rise while $\beta < 1$ slows the decay (solid blue), and $\alpha < 1$ hastens the rise while $\beta > 1$ sharpens the decay (dotted red). The core $[z_2, z_3]$ is unaffected.

Definition 3.3 (Parametrized trapezoidal neutrosophic number). A parametrized trapezoidal neutrosophic number (PTNN) is defined as

$$\tilde{N}^{(\alpha, \beta)} = \langle (t_1, t_2, t_3, t_4), (u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4); \alpha, \beta \rangle,$$

where $\alpha > 0$, $\beta > 0$, and the three quadruples satisfy

$$0 \leq t_1 < t_2 \leq t_3 < t_4 \leq 1,$$

$$0 \leq u_1 < u_2 \leq u_3 < u_4 \leq 1,$$

and

$$0 \leq v_1 < v_2 \leq v_3 < v_4 \leq 1.$$

The corresponding truth, indeterminacy, and falsity membership functions are given by

$$T^{(\alpha, \beta)}(x) = \mu_{[t_1, t_2, t_3, t_4]}^{(\alpha, \beta)}(x),$$

$$I^{(\alpha, \beta)}(x) = \mu_{[u_1, u_2, u_3, u_4]}^{(\alpha, \beta)}(x),$$

and

$$F^{(\alpha, \beta)}(x) = \mu_{[v_1, v_2, v_3, v_4]}^{(\alpha, \beta)}(x).$$

The notation (t_i) , (u_i) , and (v_i) distinguishes the truth, indeterminacy, and falsity components, respectively. The membership functions $T^{(\alpha, \beta)}(x)$, $I^{(\alpha, \beta)}(x)$, and $F^{(\alpha, \beta)}(x)$ generalise the corresponding linear trapezoidal components in Ye's formulation by incorporating the shape parameters (α, β) . When $\alpha = \beta = 1$, each parametrized membership function reduces to its

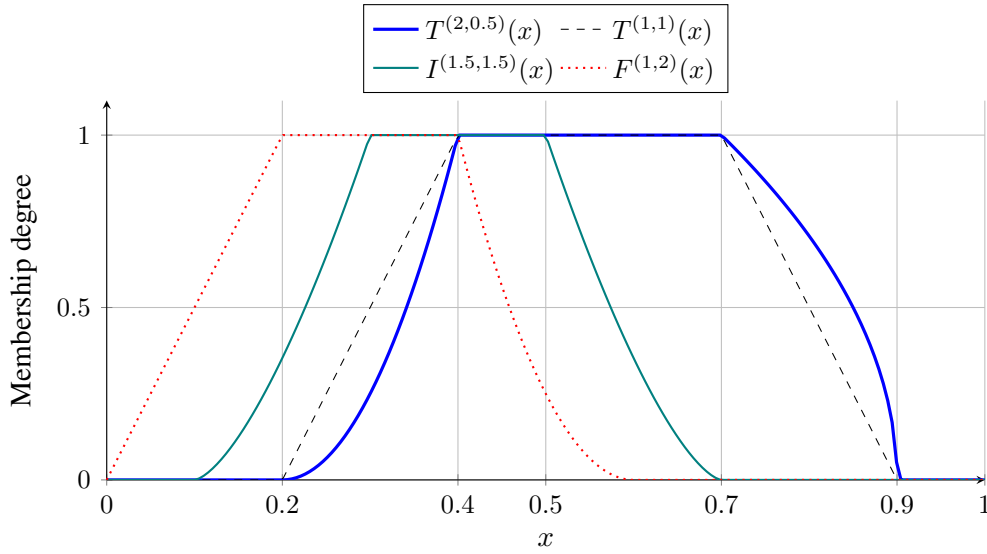


Figure 2: Illustration of a parametrized trapezoidal neutrosophic set. The universe X is represented on the horizontal axis, while the vertical axis shows the degrees of truth $T^{(\alpha_T, \beta_T)}(x)$, indeterminacy $I^{(\alpha_I, \beta_I)}(x)$, and falsity $F^{(\alpha_F, \beta_F)}(x)$ for a given element x . The solid curve $T^{(2,0.5)}(x)$ demonstrates the effect of nonlinear shape parameters $(\alpha_T, \beta_T) = (2, 0.5)$ on the truth membership, compared with the dashed linear reference $T^{(1,1)}(x)$. The functions $I^{(1.5,1.5)}(x)$ and $F^{(1,2)}(x)$ illustrate how different choices of (α, β) capture asymmetric and nonlinear behaviours in the indeterminacy and falsity components of the neutrosophic description.

classical piecewise linear trapezoidal form, and the conventional trapezoidal neutrosophic number is recovered.

Example 3.4 (Numerical illustration of Definition 3.3). Consider the parametrized trapezoidal neutrosophic number $\tilde{N}^{(\alpha, \beta)}$ defined by

$$(t_1, t_2, t_3, t_4) = (0.3, 0.5, 0.7, 0.9), \quad (u_1, u_2, u_3, u_4) = (0.0, 0.2, 0.3, 0.5), \\ (v_1, v_2, v_3, v_4) = (0.1, 0.2, 0.4, 0.6), \quad \alpha = 2, \quad \beta = 1.5.$$

According to Definition 3.3, the corresponding truth, indeterminacy, and falsity membership functions are

$$T^{(2,1.5)}(x) = \mu_{[0.3, 0.5, 0.7, 0.9]}^{(2,1.5)}(x),$$

$$I^{(2,1.5)}(x) = \mu_{[0.0, 0.2, 0.3, 0.5]}^{(2,1.5)}(x),$$

and

$$F^{(2,1.5)}(x) = \mu_{[0.1, 0.2, 0.4, 0.6]}^{(2,1.5)}(x).$$

We evaluate the three membership degrees at $x = 0.45$.

Truth membership. For $(t_1, t_2, t_3, t_4) = (0.3, 0.5, 0.7, 0.9)$, we have $0.3 < 0.45 \leq 0.5$, so $x = 0.45$ lies in the ascending region. Therefore,

$$T^{(2,1.5)}(0.45) = \left(\frac{0.45 - 0.3}{0.5 - 0.3} \right)^2 = \left(\frac{0.15}{0.2} \right)^2 = (0.75)^2 = 0.5625.$$

Indeterminacy membership. For $(u_1, u_2, u_3, u_4) = (0.0, 0.2, 0.3, 0.5)$, we have $0.3 \leq 0.45 < 0.5$, so $x = 0.45$ lies in the descending region. Hence,

$$I^{(2,1.5)}(0.45) = \left(\frac{0.5 - 0.45}{0.5 - 0.3} \right)^{1.5} = \left(\frac{0.05}{0.2} \right)^{1.5} = (0.25)^{1.5} = 0.125.$$

Falsity membership. For $(v_1, v_2, v_3, v_4) = (0.1, 0.2, 0.4, 0.6)$, we have $0.4 \leq 0.45 < 0.6$, so $x = 0.45$ also lies in the descending region. Thus,

$$F^{(2,1.5)}(0.45) = \left(\frac{0.6 - 0.45}{0.6 - 0.4} \right)^{1.5} = \left(\frac{0.15}{0.2} \right)^{1.5} = (0.75)^{1.5} \approx 0.6495.$$

Hence, at $x = 0.45$, the corresponding neutrosophic membership triple is approximately

$$(T^{(2,1.5)}(0.45), I^{(2,1.5)}(0.45), F^{(2,1.5)}(0.45)) = (0.5625, 0.1250, 0.6495).$$

Thus, at $x = 0.45$, the parametrized trapezoidal neutrosophic number encodes the triple

$$(T^{(2,1.5)}(0.45), I^{(2,1.5)}(0.45), F^{(2,1.5)}(0.45)) \approx (0.5625, 0.125, 0.6495),$$

demonstrating that the neutrosophic components need not sum to one and can simultaneously capture moderate truth, low indeterminacy, and relatively high falsity, in line with the general neutrosophic philosophy.

3.3. Parametric Score and Accuracy Functions

To rank parametrized trapezoidal neutrosophic numbers, a scalar representative is required for each trapezoidal component. In the classical formulation of Ye (2015), the four vertices are aggregated using their arithmetic mean. Although computationally simple, this representation does not account for the shape of the membership function. Consequently, two components with identical vertices but different curvatures would produce the same representative value if the shape parameters were not incorporated separately.

In the proposed framework, each component is instead represented by the *centroid* of its parametrized membership function. The shape parameters therefore influence the representative value through the geometry of the membership function itself, without requiring auxiliary exponentiation. Moreover, the truth, indeterminacy, and falsity components are treated consistently using the same representative-value operator.

Definition 3.5 (Representative value). Let

$$z = (z_1, z_2, z_3, z_4)$$

satisfy

$$0 \leq z_1 < z_2 \leq z_3 < z_4 \leq 1,$$

and let $\mu_{[z_1, z_2, z_3, z_4]}^{(\alpha, \beta)}$ be the parametrized trapezoidal membership function defined in Defini-

tion 3.1, where $\alpha, \beta > 0$. The representative value of z is defined as the centroid

$$V(z; \alpha, \beta) = \frac{\int_{z_1}^{z_4} x \mu_{[z_1, z_2, z_3, z_4]}^{(\alpha, \beta)}(x) dx}{\int_{z_1}^{z_4} \mu_{[z_1, z_2, z_3, z_4]}^{(\alpha, \beta)}(x) dx} = \frac{M(z; \alpha, \beta)}{A(z; \alpha, \beta)}. \quad (2)$$

For the degenerate singleton case $z_1 = z_2 = z_3 = z_4$, we define

$$V(z; \alpha, \beta) := z_1$$

by continuous extension.

Evaluating the integrals over the ascending segment $[z_1, z_2]$, the core $[z_2, z_3]$, and the descending segment $[z_3, z_4]$ gives

$$A(z; \alpha, \beta) = \frac{z_2 - z_1}{\alpha + 1} + (z_3 - z_2) + \frac{z_4 - z_3}{\beta + 1}, \quad (3)$$

$$\begin{aligned} M(z; \alpha, \beta) &= (z_2 - z_1) \left[\frac{z_1}{\alpha + 1} + \frac{z_2 - z_1}{\alpha + 2} \right] + \frac{z_3^2 - z_2^2}{2} \\ &\quad + (z_4 - z_3) \left[\frac{z_4}{\beta + 1} - \frac{z_4 - z_3}{\beta + 2} \right]. \end{aligned} \quad (4)$$

Since $\alpha, \beta > 0$ and $z_1 < z_4$, we have $A(z; \alpha, \beta) > 0$. Hence, the representative value in Eq. 2 is well defined. The singleton convention is obtained as the continuous limit when the support width $z_4 - z_1$ tends to zero.

Lemma 3.6 (Reduction at $(\alpha, \beta) = (1, 1)$). *When $(\alpha, \beta) = (1, 1)$, the representative value becomes*

$$V(z; 1, 1) = \frac{z_1^2 + z_1 z_2 + z_2^2 - z_3^2 - z_3 z_4 - z_4^2}{3(z_1 + z_2 - z_3 - z_4)}, \quad (5)$$

which is the centroid of the classical piecewise linear trapezoidal membership function.

If the trapezoid is symmetric, that is,

$$z_2 - z_1 = z_4 - z_3,$$

equivalently,

$$z_1 + z_4 = z_2 + z_3,$$

then

$$V(z; 1, 1) = \frac{1}{4}(z_1 + z_2 + z_3 + z_4),$$

which coincides with the vertex mean used in the classical formulation of Ye (2015).

Proof. Substituting $\alpha = \beta = 1$ into Eqs. 3 and 4 and simplifying the ratio M/A gives Eq. 5.

For a symmetric trapezoid, $\mu_{[z_1, z_2, z_3, z_4]}^{(1,1)}$ is symmetric about

$$c = \frac{z_1 + z_4}{2} = \frac{z_2 + z_3}{2}.$$

Hence, its centroid is c . Since $z_1 + z_4 = z_2 + z_3$, it follows that

$$c = \frac{z_1 + z_2 + z_3 + z_4}{4},$$

which proves the result. $\square$

For a parametrized trapezoidal neutrosophic number

$$\tilde{N}^{(\alpha, \beta)} = \langle t, u, v; \alpha, \beta \rangle,$$

let

$$V_T = V(t; \alpha, \beta), \quad V_I = V(u; \alpha, \beta), \quad V_F = V(v; \alpha, \beta)$$

denote the representative values of its truth, indeterminacy, and falsity components, respectively.

Definition 3.7 (Parametric score function). The parametric score of $\tilde{N}^{(\alpha, \beta)}$ is defined as

$$S_{\alpha, \beta}(\tilde{N}^{(\alpha, \beta)}) = \frac{1}{3} [V_T + (1 - V_I) + (1 - V_F)]. \quad (6)$$

Definition 3.8 (Parametric accuracy function). The parametric accuracy of $\tilde{N}^{(\alpha, \beta)}$ is defined as

$$H_{\alpha, \beta}(\tilde{N}^{(\alpha, \beta)}) = V_T - V_F. \quad (7)$$

Three features of the score function in Eq. 6 are deliberate and address recurring issues in neutrosophic scoring. First, all three components are evaluated using the same representative-value operator $V(\cdot; \alpha, \beta)$. Consequently, indeterminacy is defuzzified in the same manner as truth and falsity, avoiding an asymmetric or ad hoc treatment of any individual component. Second, truth contributes positively, whereas indeterminacy and falsity enter through their complements, $1 - V_I$ and $1 - V_F$, respectively. This construction ensures that the score lies in $[0, 1]$ and is oriented such that a larger value represents a more preferable alternative, as established in Theorem 3.11. Third, the additive structure provides a centroid-based analogue of the classical score formulation. When $(\alpha, \beta) = (1, 1)$, it is based on the centroids of the classical linear trapezoidal components and coincides with the corresponding vertex-mean formulation whenever those components are symmetric, as described in Lemma 3.6. The additive form is retained rather than replaced by a multiplicative or distance-based formulation so that the effect of the proposed parametrization can be examined independently in Section 4.

Remark 3.9 (Practical selection of (α, β)). The setting $(\alpha, \beta) = (1, 1)$ provides the natural neutral baseline because it recovers the classical piecewise linear trapezoidal membership geometry. For symmetric component trapezoids, the resulting centroid also coincides with the classical vertex mean described in Lemma 3.6. When no prior information regarding membership curvature is available, this setting therefore provides an appropriate reference configuration.

Departures from $(1, 1)$ represent nonlinear interpretations of the underlying assessments. In particular, $\alpha > 1$ produces a more conservative increase in membership over the ascending segment, whereas $\alpha < 1$ produces a more rapid initial increase. Similarly, $\beta > 1$ produces a more rapid initial decrease over the descending segment, whereas $\beta < 1$ produces a slower initial decrease followed by a sharper decline near the end of the segment.

In practical applications, results should preferably be reported over a suitable range of (α, β) values, as illustrated in the sensitivity analysis in Section 4.3, rather than relying exclusively on a single non-neutral parameter pair. When specific values of (α, β) are selected, they should be

justified by the decision context, for example, through the manner in which increasing support or decreasing opposition is interpreted. The same parameter values should also be applied consistently across alternatives to prevent the parametrization itself from favouring any particular alternative.

Example 3.10 (Numerical illustration of the parametric score and accuracy). We continue with the parametrized trapezoidal neutrosophic number $\tilde{N}^{(\alpha,\beta)}$ from Example 3.4, namely,

$$\begin{aligned}(t_1, t_2, t_3, t_4) &= (0.3, 0.5, 0.7, 0.9), & (u_1, u_2, u_3, u_4) &= (0.0, 0.2, 0.3, 0.5), \\ (v_1, v_2, v_3, v_4) &= (0.1, 0.2, 0.4, 0.6), & \alpha &= 2, \quad \beta = 1.5.\end{aligned}$$

Using Eqs. 3 and 4, the representative value of each component is obtained from its centroid. For the truth component,

$$A(t; 2, 1.5) = \frac{0.2}{3} + 0.2 + \frac{0.2}{2.5} = 0.3467,$$

and hence

$$V_T = V(t; 2, 1.5) \approx 0.6074.$$

Applying the same procedure to the indeterminacy and falsity components gives

$$V_I = V(u; 2, 1.5) \approx 0.2577, \quad V_F = V(v; 2, 1.5) \approx 0.3268.$$

By Definitions 3.7 and 3.8, the parametric score is

$$S_{2,1.5} \left(\tilde{N}^{(2,1.5)} \right) = \frac{1}{3} [0.6074 + (1 - 0.2577) + (1 - 0.3268)] \approx 0.6743,$$

while the corresponding parametric accuracy is

$$H_{2,1.5} \left(\tilde{N}^{(2,1.5)} \right) = 0.6074 - 0.3268 \approx 0.2806.$$

For comparison, when $(\alpha, \beta) = (1, 1)$, the representative values are

$$V_T = 0.6000, \quad V_I = 0.2500, \quad V_F \approx 0.3286.$$

These values yield

$$S_{1,1} \approx 0.6738, \quad H_{1,1} \approx 0.2714.$$

The truth and indeterminacy components in this example are symmetric. Hence, by Lemma 3.6, their $(1, 1)$ centroids coincide with their corresponding vertex means, namely 0.6000 and 0.2500. The falsity component is asymmetric, so its centroid, approximately 0.3286, differs slightly from its vertex mean of 0.3250.

The change from $(\alpha, \beta) = (1, 1)$ to $(2, 1.5)$ produces only a modest change in both the score and accuracy values. This illustrates how the proposed parametrization can refine the evaluation by incorporating membership curvature without causing an abrupt change in the underlying assessment.

3.4. Theoretical Properties

We now establish several desirable properties of the proposed ranking framework for parametrized trapezoidal neutrosophic numbers. In particular, we show that the representative value and the associated score and accuracy functions remain within their admissible ranges, respond consistently to changes in the underlying data, vary smoothly with the shape parameters, and exhibit predictable limiting behaviour. Throughout this subsection, the component quadruples are assumed to satisfy the admissibility conditions stated in Definition 3.5, unless otherwise specified. The singleton case is covered by the continuous-extension convention in that definition.

Theorem 3.11 (Boundedness and orientation). *For every parametrized trapezoidal neutrosophic number and all $\alpha, \beta > 0$,*

$$V(z; \alpha, \beta) \in [z_1, z_4] \subseteq [0, 1], \quad S_{\alpha, \beta} \in [0, 1], \quad H_{\alpha, \beta} \in [-1, 1].$$

Proof. The representative value $V(z; \alpha, \beta)$ is a weighted average of x with respect to the non-negative membership function $\mu_{[z_1, z_2, z_3, z_4]}^{(\alpha, \beta)}$ supported on $[z_1, z_4]$. Therefore,

$$V(z; \alpha, \beta) \in [z_1, z_4] \subseteq [0, 1].$$

Consequently,

$$V_T, V_I, V_F \in [0, 1].$$

Each term in the score function,

$$V_T, \quad 1 - V_I, \quad 1 - V_F,$$

belongs to $[0, 1]$. Hence, their arithmetic mean satisfies

$$S_{\alpha, \beta} \in [0, 1].$$

Similarly,

$$H_{\alpha, \beta} = V_T - V_F$$

is the difference of two quantities in $[0, 1]$, and therefore

$$H_{\alpha, \beta} \in [-1, 1]. \quad \square$$

Proposition 3.12 (Monotonicity with respect to the data). For fixed $\alpha, \beta > 0$, the representative value $V(z; \alpha, \beta)$ is non-decreasing with respect to each vertex z_1, z_2, z_3, z_4 , provided that the perturbation preserves the admissible ordering of the vertices. Under a non-degenerate perturbation, the increase is strict.

Consequently, the score $S_{\alpha, \beta}$ is non-decreasing with respect to the truth-component vertices and non-increasing with respect to the indeterminacy- and falsity-component vertices.

Proof. Increasing any one of the vertices while preserving the admissible ordering shifts the corresponding parametrized membership distribution towards larger values of x . In terms of the normalised membership density

$$\frac{\mu_{[z_1, z_2, z_3, z_4]}^{(\alpha, \beta)}(x)}{A(z; \alpha, \beta)},$$

such a change induces a rightward shift in the distribution and therefore cannot decrease its mean. Hence,

$$V(z; \alpha, \beta)$$

is non-decreasing with respect to each vertex, with strict increase whenever a positive amount of membership mass is shifted to the right.

The corresponding statement for the score follows directly from Eq. 6. The representative value V_T enters with a positive sign, whereas V_I and V_F enter with negative signs through $1 - V_I$ and $1 - V_F$, respectively. Therefore, increasing the truth vertices cannot decrease the score, while increasing the indeterminacy or falsity vertices cannot increase it. $\square$

Proposition 3.13 (Smoothness and directional response to the shape parameters). For every admissible non-degenerate component z , $V(z; \alpha, \beta)$ is C^∞ on the open quadrant

$$\{(\alpha, \beta) : \alpha > 0, \beta > 0\}.$$

Consequently, $S_{\alpha, \beta}$ and $H_{\alpha, \beta}$ are also C^∞ functions of (α, β) .

Moreover,

$$\frac{\partial V}{\partial \alpha} \geq 0, \quad \frac{\partial V}{\partial \beta} \leq 0,$$

with strict inequalities when the ascending and descending segments, respectively, have positive width.

Proof. From Eqs. 3 and 4, both $A(z; \alpha, \beta)$ and $M(z; \alpha, \beta)$ are composed of rational functions involving the factors

$$\alpha + 1, \quad \alpha + 2, \quad \beta + 1, \quad \beta + 2.$$

None of these factors vanishes for $\alpha, \beta > 0$, and

$$A(z; \alpha, \beta) > 0.$$

Hence,

$$V(z; \alpha, \beta) = \frac{M(z; \alpha, \beta)}{A(z; \alpha, \beta)}$$

is infinitely differentiable on the open positive quadrant. Since $S_{\alpha, \beta}$ and $H_{\alpha, \beta}$ are formed from linear combinations of the representative values, they are also C^∞ .

For the ascending segment, increasing α decreases the membership values on the interior of $[z_1, z_2]$ more strongly near z_1 than near z_2 . The resulting normalised membership mass is therefore shifted towards larger values of x , implying

$$\frac{\partial V}{\partial \alpha} \geq 0,$$

with strict inequality when $z_1 < z_2$.

Similarly, increasing β concentrates the descending-segment membership mass towards z_3 , thereby shifting that portion of the distribution towards smaller values of x . Hence,

$$\frac{\partial V}{\partial \beta} \leq 0,$$

with strict inequality when $z_3 < z_4$. $\square$

It should be noted that the componentwise monotonicity of V with respect to α and β does not imply that the overall score $S_{\alpha,\beta}$ is necessarily monotone in either parameter. Indeed,

$$\frac{\partial S_{\alpha,\beta}}{\partial \alpha} = \frac{1}{3} \left(\frac{\partial V_T}{\partial \alpha} - \frac{\partial V_I}{\partial \alpha} - \frac{\partial V_F}{\partial \alpha} \right),$$

and an analogous expression holds for $\partial S_{\alpha,\beta}/\partial \beta$. The resulting sign therefore depends on the relative responses of the truth, indeterminacy, and falsity components.

Proposition 3.14 (Limiting behaviour). For fixed admissible z ,

$$\lim_{\alpha \rightarrow \infty} V(z; \alpha, \beta) = \frac{\frac{1}{2}(z_3^2 - z_2^2) + (z_4 - z_3) \left[\frac{z_4}{\beta + 1} - \frac{z_4 - z_3}{\beta + 2} \right]}{(z_3 - z_2) + \frac{z_4 - z_3}{\beta + 1}}.$$

Thus, as $\alpha \rightarrow \infty$, the contribution of the ascending segment vanishes and the centroid is determined by the core and descending segments.

The corresponding result for $\beta \rightarrow \infty$ is obtained analogously by suppressing the descending-segment contribution.

Furthermore,

$$\lim_{\alpha, \beta \rightarrow 0^+} V(z; \alpha, \beta) = \frac{z_1 + z_4}{2}.$$

Proof. Taking the limit $\alpha \rightarrow \infty$ in Eqs. 3 and 4 gives

$$\frac{1}{\alpha + 1} \rightarrow 0, \quad \frac{1}{\alpha + 2} \rightarrow 0.$$

Therefore, all contributions associated with the ascending segment vanish, leaving the stated expression. The case $\beta \rightarrow \infty$ follows analogously.

As $\alpha \rightarrow 0^+$,

$$\left(\frac{x - z_1}{z_2 - z_1} \right)^\alpha \rightarrow 1 \quad \text{for } z_1 < x \leq z_2,$$

while as $\beta \rightarrow 0^+$,

$$\left(\frac{z_4 - x}{z_4 - z_3} \right)^\beta \rightarrow 1 \quad \text{for } z_3 \leq x < z_4.$$

Together with the unit membership on the core interval, the parametrized membership function therefore converges almost everywhere to the indicator function of $[z_1, z_4]$. Its centroid is the midpoint of the interval, namely

$$\frac{z_1 + z_4}{2}. \quad \square$$

Theorem 3.15 (Backward compatibility and ranking consistency). When $(\alpha, \beta) = (1, 1)$, the ordering induced by $(S_{1,1}, H_{1,1})$ through Definition 3.17 coincides with the classical trape-

zoidal neutrosophic ordering whenever the truth, indeterminacy, and falsity components are symmetric. For general trapezoidal components, the resulting ordering is based on their classical piecewise linear centroids.

For the decision matrix considered in Section 4, the SYAI ranking obtained under $(\alpha, \beta) = (1, 1)$ is numerically identical to that obtained using Ye's vertex-mean formulation, as reported in Table 3.

Proof. For symmetric components, Lemma 3.6 gives

$$V(z; 1, 1) = \frac{1}{4} \sum_{i=1}^4 z_i.$$

Hence, the representative values used in $S_{1,1}$ and $H_{1,1}$ coincide with the corresponding vertex means in Ye's formulation. Therefore, the resulting score and accuracy values, and consequently the induced ordering, are identical.

For general trapezoidal components, $V(z; 1, 1)$ is given by the centroid in Eq. 5. Thus, $(S_{1,1}, H_{1,1})$ provides the centroid-based analogue of the classical score–accuracy pair. For the decision matrix analysed in Section 4, the numerical comparison reported in Table 3 confirms that this centroid-based formulation produces the same SYAI ranking as the vertex-mean formulation. $\square$

Remark 3.16 (Status of the arithmetic operations). The operations $\oplus$, $\otimes$, $\odot$, and the power operation introduced in Section 3.6 act on the trapezoidal vertices in the same manner as in the formulation of Ye (2015), while the parameters (α, β) are carried through unchanged.

This construction distinguishes between the algebraic and geometric roles of the parametrization. The parameters govern the *representation* of each number through its membership geometry and influence its *ranking* through the score function in Definition 3.7. They do not, however, alter the underlying vertex-level arithmetic, which is retained as a closure-preserving operation on the trapezoidal structure.

Accordingly, the closure results in Theorems 3.20 and 3.22 follow from the corresponding vertex-level operations, whereas the modelling contribution of the parametrization lies in the nonlinear membership geometry and the resulting centroid-based score. This separation is deliberate: it preserves algebraic tractability while allowing the effect of the shape parameters on the ranking process to be examined independently in Section 4.

3.5. Ordering of Parametrized Trapezoidal Neutrosophic Numbers

Definition 3.17 (Order relation). Let $\tilde{N}_1^{(\alpha, \beta)}$ and $\tilde{N}_2^{(\alpha, \beta)}$ be two parametrized trapezoidal neutrosophic numbers defined under the same shape parameters (α, β) .

We write

$$\tilde{N}_1^{(\alpha, \beta)} \succ \tilde{N}_2^{(\alpha, \beta)}$$

if

$$S_{\alpha, \beta} \left(\tilde{N}_1^{(\alpha, \beta)} \right) > S_{\alpha, \beta} \left(\tilde{N}_2^{(\alpha, \beta)} \right).$$

If

$$S_{\alpha, \beta} \left(\tilde{N}_1^{(\alpha, \beta)} \right) = S_{\alpha, \beta} \left(\tilde{N}_2^{(\alpha, \beta)} \right),$$

then the accuracy function is used as a tie-breaker, so that

$$H_{\alpha,\beta} \left(\tilde{N}_1^{(\alpha,\beta)} \right) > H_{\alpha,\beta} \left(\tilde{N}_2^{(\alpha,\beta)} \right)$$

implies

$$\tilde{N}_1^{(\alpha,\beta)} \succ \tilde{N}_2^{(\alpha,\beta)}.$$

Definition 3.17 parallels the comparison principle in Ye's classical formulation, while replacing the classical score and accuracy measures with their parametric counterparts. This provides a consistent ordering mechanism within the proposed nonlinear framework.

Example 3.18 (Numerical ordering of parametrized trapezoidal neutrosophic numbers). We illustrate Definition 3.17 by comparing two parametrized trapezoidal neutrosophic numbers under the common shape parameters $(\alpha, \beta) = (2, 1.5)$. Consider

$$\tilde{N}_1^{(2,1.5)} = \left\langle (0.3, 0.5, 0.7, 0.9), (0.0, 0.2, 0.3, 0.5), (0.1, 0.2, 0.4, 0.6); 2, 1.5 \right\rangle,$$

and

$$\tilde{N}_2^{(2,1.5)} = \left\langle (0.5, 0.6, 0.8, 0.9), (0.1, 0.2, 0.3, 0.4), (0.05, 0.15, 0.25, 0.5); 2, 1.5 \right\rangle.$$

For $\tilde{N}_1^{(2,1.5)}$, the representative values obtained in Example 3.10 are

$$V_T^{(1)} = 0.6074, \quad V_I^{(1)} = 0.2577, \quad V_F^{(1)} = 0.3268.$$

Hence,

$$S_{2,1.5} \left(\tilde{N}_1^{(2,1.5)} \right) \approx 0.6743,$$

and

$$H_{2,1.5} \left(\tilde{N}_1^{(2,1.5)} \right) \approx 0.2806.$$

For $\tilde{N}_2^{(2,1.5)}$, applying Eqs. 3 and 4 to each component gives

$$V_T^{(2)} = 0.7036, \quad V_I^{(2)} = 0.2537, \quad V_F^{(2)} = 0.2413.$$

Therefore,

$$S_{2,1.5} \left(\tilde{N}_2^{(2,1.5)} \right) = \frac{1}{3} [0.7036 + (1 - 0.2537) + (1 - 0.2413)] \approx 0.7362,$$

while

$$H_{2,1.5} \left(\tilde{N}_2^{(2,1.5)} \right) = 0.7036 - 0.2413 \approx 0.4622.$$

Ordering decision. Since

$$S_{2,1.5} \left(\tilde{N}_2^{(2,1.5)} \right) \approx 0.7362 > 0.6743 \approx S_{2,1.5} \left(\tilde{N}_1^{(2,1.5)} \right),$$

Definition 3.17 yields

$$\tilde{N}_2^{(2,1.5)} \succ \tilde{N}_1^{(2,1.5)}.$$

The comparison is therefore resolved at the score level. The accuracy values are not required in this case; if the scores were equal, the alternative with the larger accuracy value would be preferred according to Definition 3.17.

3.6. Arithmetic Operations of Parametrized Trapezoidal Neutrosophic Numbers

In this subsection, we extend the arithmetic operations for trapezoidal neutrosophic numbers introduced by Ye (2015) to the parametrized setting. The operations act on the trapezoidal vertices of the truth, indeterminacy, and falsity components in the same manner as in the classical formulation, while the shape parameters (α, β) are preserved. Consequently, the operations modify the underlying trapezoidal structure without altering the curvature parameters associated with the corresponding membership functions.

Let

$$\tilde{N}_1^{(\alpha, \beta)} = \langle (t_{11}, t_{12}, t_{13}, t_{14}), (u_{11}, u_{12}, u_{13}, u_{14}), (v_{11}, v_{12}, v_{13}, v_{14}); \alpha, \beta \rangle$$

and

$$\tilde{N}_2^{(\alpha, \beta)} = \langle (t_{21}, t_{22}, t_{23}, t_{24}), (u_{21}, u_{22}, u_{23}, u_{24}), (v_{21}, v_{22}, v_{23}, v_{24}); \alpha, \beta \rangle$$

be two parametrized trapezoidal neutrosophic numbers.

Here, the first subscript identifies the number, $k = 1, 2$, while the second identifies the trapezoidal vertex, $i = 1, 2, 3, 4$. Thus, t_{ki} , u_{ki} , and v_{ki} denote the i th vertices of the truth, indeterminacy, and falsity components of $\tilde{N}_k^{(\alpha, \beta)}$, respectively. For $k = 1, 2$, the component quadruples satisfy

$$0 \leq t_{k1} < t_{k2} \leq t_{k3} < t_{k4} \leq 1,$$

$$0 \leq u_{k1} < u_{k2} \leq u_{k3} < u_{k4} \leq 1,$$

and

$$0 \leq v_{k1} < v_{k2} \leq v_{k3} < v_{k4} \leq 1,$$

where $\alpha > 0$ and $\beta > 0$ are common shape parameters.

3.6.1. Addition and multiplication

Following the operational rules for trapezoidal neutrosophic numbers given by Ye (2015), we define the addition and multiplication of two parametrized trapezoidal neutrosophic numbers as follows.

Definition 3.19 (Addition and multiplication). The sum $\tilde{N}_1^{(\alpha, \beta)} \oplus \tilde{N}_2^{(\alpha, \beta)}$ is defined by

$$\begin{aligned} \tilde{N}_1^{(\alpha, \beta)} \oplus \tilde{N}_2^{(\alpha, \beta)} = \langle (t'_1, t'_2, t'_3, t'_4), (u'_1, u'_2, u'_3, u'_4), \\ (v'_1, v'_2, v'_3, v'_4); \alpha, \beta \rangle, \end{aligned} \quad (8)$$

where

$$t'_i = t_{1i} + t_{2i} - t_{1i}t_{2i}, \quad u'_i = u_{1i}u_{2i}, \quad v'_i = v_{1i}v_{2i}, \quad i = 1, 2, 3, 4.$$

The product $\tilde{N}_1^{(\alpha, \beta)} \otimes \tilde{N}_2^{(\alpha, \beta)}$ is defined by

$$\begin{aligned} \tilde{N}_1^{(\alpha, \beta)} \otimes \tilde{N}_2^{(\alpha, \beta)} = \left\langle (\hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{t}_4), (\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4), \right. \\ \left. (\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4); \alpha, \beta \right\rangle, \end{aligned} \quad (9)$$

where

$$\hat{t}_i = t_{1i}t_{2i}, \quad \hat{u}_i = u_{1i} + u_{2i} - u_{1i}u_{2i}, \quad \hat{v}_i = v_{1i} + v_{2i} - v_{1i}v_{2i}, \quad i = 1, 2, 3, 4.$$

Theorem 3.20 (Closure under addition and multiplication). *The addition and multiplication operations in Definition 3.19 are closed within the class of parametrized trapezoidal neutrosophic numbers. Specifically,*

$$\tilde{N}_1^{(\alpha, \beta)} \oplus \tilde{N}_2^{(\alpha, \beta)}$$

and

$$\tilde{N}_1^{(\alpha, \beta)} \otimes \tilde{N}_2^{(\alpha, \beta)}$$

satisfy the admissibility conditions of Definition 3.3 and retain the common shape parameters (α, β) .

Proof. Consider first the vertex-level transformations in Eqs. 8 and 9. For $x, y \in [0, 1]$, define

$$P(x, y) = xy, \quad Q(x, y) = x + y - xy.$$

Both P and Q map $[0, 1]^2$ into $[0, 1]$ and are non-decreasing in each argument. Consequently, applying these functions componentwise preserves the ordering of the trapezoidal vertices.

For the addition operation,

$$t'_i = Q(t_{1i}, t_{2i}), \quad u'_i = P(u_{1i}, u_{2i}), \quad v'_i = P(v_{1i}, v_{2i}),$$

while for the multiplication operation,

$$\hat{t}_i = P(t_{1i}, t_{2i}), \quad \hat{u}_i = Q(u_{1i}, u_{2i}), \quad \hat{v}_i = Q(v_{1i}, v_{2i}).$$

Hence, all transformed vertices remain in $[0, 1]$ and retain their required trapezoidal ordering. These are precisely the vertex-level operations used in the classical trapezoidal neutrosophic formulation of Ye (2015).

Since the common shape parameters (α, β) are carried through unchanged, the resulting triples satisfy the structural requirements of Definition 3.3. Therefore,

$$\tilde{N}_1^{(\alpha, \beta)} \oplus \tilde{N}_2^{(\alpha, \beta)}$$

and

$$\tilde{N}_1^{(\alpha, \beta)} \otimes \tilde{N}_2^{(\alpha, \beta)}$$

are parametrized trapezoidal neutrosophic numbers. This establishes closure. $\square$

The operations $\oplus$ and $\otimes$ are commutative and associative because the underlying functions

$$xy \quad \text{and} \quad x + y - xy$$

are commutative and associative on $[0, 1]$. Since the shape parameters (α, β) remain unchanged, these properties extend directly to the corresponding PTNN operations.

3.6.2. Scalar multiplication and power

We next define scalar multiplication and power operations for PTNNs, following the corresponding vertex-level structure of Ye (2015).

Definition 3.21 (Scalar multiplication and power). Let $k > 0$ be a real scalar. The *scalar multiple* of $\tilde{N}_1^{(\alpha, \beta)}$ by k is defined as

$$k \odot \tilde{N}_1^{(\alpha, \beta)} = \left\langle (t_1^{(k)}, t_2^{(k)}, t_3^{(k)}, t_4^{(k)}), (u_1^{(k)}, u_2^{(k)}, u_3^{(k)}, u_4^{(k)}), (v_1^{(k)}, v_2^{(k)}, v_3^{(k)}, v_4^{(k)}); \alpha, \beta \right\rangle, \quad (10)$$

where

$$t_i^{(k)} = 1 - (1 - t_{1i})^k, \quad u_i^{(k)} = (u_{1i})^k, \quad v_i^{(k)} = (v_{1i})^k, \quad i = 1, 2, 3, 4.$$

Similarly, the *kth power* of $\tilde{N}_1^{(\alpha, \beta)}$ is defined by

$$(\tilde{N}_1^{(\alpha, \beta)})^{*k} = \left\langle (\tilde{t}_1^{(k)}, \tilde{t}_2^{(k)}, \tilde{t}_3^{(k)}, \tilde{t}_4^{(k)}), (\tilde{u}_1^{(k)}, \tilde{u}_2^{(k)}, \tilde{u}_3^{(k)}, \tilde{u}_4^{(k)}), (\tilde{v}_1^{(k)}, \tilde{v}_2^{(k)}, \tilde{v}_3^{(k)}, \tilde{v}_4^{(k)}); \alpha, \beta \right\rangle, \quad (11)$$

where

$$\tilde{t}_i^{(k)} = (t_{1i})^k, \quad \tilde{u}_i^{(k)} = 1 - (1 - u_{1i})^k, \quad \tilde{v}_i^{(k)} = 1 - (1 - v_{1i})^k, \quad i = 1, 2, 3, 4.$$

Theorem 3.22 (Closure under scalar multiplication and power). For every $k > 0$,

$$k \odot \tilde{N}_1^{(\alpha, \beta)}$$

and

$$(\tilde{N}_1^{(\alpha, \beta)})^{*k}$$

are parametrized trapezoidal neutrosophic numbers.

Proof. For $k > 0$, the functions

$$f_k(x) = x^k$$

and

$$g_k(x) = 1 - (1 - x)^k$$

map $[0, 1]$ into $[0, 1]$ and are strictly increasing on this interval. Consequently, applying either transformation componentwise preserves the ordering of the vertices of every non-degenerate trapezoidal component.

In Eq. 10, the truth vertices are transformed using g_k , whereas the indeterminacy and falsity vertices are transformed using f_k . In Eq. 11, the truth vertices are transformed using f_k , whereas the indeterminacy and falsity vertices are transformed using g_k . Hence, in both cases, all resulting vertices remain in $[0, 1]$ and preserve the required trapezoidal ordering.

These transformations coincide with the corresponding vertex-level operations in the classical trapezoidal neutrosophic formulation of Ye (2015). Since the shape parameters (α, β) are retained unchanged, both resulting objects satisfy Definition 3.3. Therefore, the class of PTNNs is closed under scalar multiplication and the power operation. $\square$

The arithmetic operations introduced above retain the vertex-level algebra of the classical trapezoidal neutrosophic framework while preserving the shape parameters (α, β) . In particular, $\oplus$ and $\otimes$ inherit commutativity and associativity from their underlying componentwise operations. The role of (α, β) remains confined to the membership geometry and the resulting centroid-based evaluation, rather than altering the vertex-level arithmetic itself.

4. MADM under the Proposed PTNS–SYAI Framework

This section presents an illustrative example adapted from the trapezoidal neutrosophic multi-attribute decision-making problem of Ye (2015). The objective is to demonstrate the implementation of the proposed parametrized trapezoidal neutrosophic set (PTNS) within the SYAI ranking framework and to examine how the shape parameters (α, β) influence the resulting evaluation and ranking. The overall PTNS–SYAI procedure is illustrated in Figure 3.

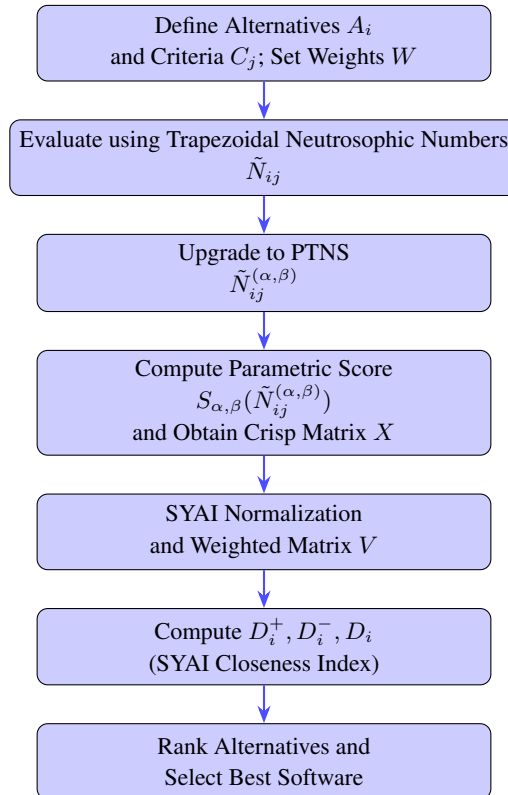


Figure 3: Workflow of the Proposed PTNS–SYAI Decision-Making Approach.

4.1. Numerical Example

An investment company evaluates five candidate software systems,

$$A = \{A_1, A_2, A_3, A_4, A_5\},$$

with respect to the following four attributes:

- C_1 : contribution to organisational performance (benefit);
- C_2 : effort required to transition from the current system (cost);
- C_3 : hardware and software investment cost (cost);
- C_4 : reliability of the outsourced developer (benefit).

The expert assessment of each alternative A_i with respect to attribute C_j is expressed as a trapezoidal neutrosophic number

$$\tilde{N}_{ij} = \langle (t_{1ij}, t_{2ij}, t_{3ij}, t_{4ij}), (u_{1ij}, u_{2ij}, u_{3ij}, u_{4ij}), (v_{1ij}, v_{2ij}, v_{3ij}, v_{4ij}) \rangle.$$

The attribute weights, elicited from the decision-makers, are given by $W = (0.25, 0.25, 0.30, 0.20)^T$, where $\sum_{j=1}^4 w_j = 1$. Among the four attributes, investment cost (C_3) receives the largest weight. The complete trapezoidal neutrosophic decision matrix is presented in Table 2 using the quad-layout representation.

Following Section 3, each entry of the decision matrix is converted into a crisp value using the parametric score function in Definition 3.7. The four attributes are classified as either benefit or cost criteria. Specifically, C_1 (contribution to organisational performance) and C_4 (developer reliability) are benefit criteria, for which larger values are preferred, whereas C_2 (transition effort) and C_3 (investment cost) are cost criteria, for which smaller values are preferred. This classification determines the criterion-specific ideal values used in the normalisation procedure described in Section 4.2.

For the baseline analysis, we set $(\alpha, \beta) = (1, 1)$. Consider alternative A_1 under criterion C_1 :

$$\tilde{N}_{11} = \langle (0.4, 0.5, 0.6, 0.7), (0.0, 0.1, 0.2, 0.3), (0.1, 0.1, 0.1, 0.1) \rangle.$$

The truth component $(0.4, 0.5, 0.6, 0.7)$ and the indeterminacy component $(0.0, 0.1, 0.2, 0.3)$ are symmetric. Hence, by Lemma 3.6, their centroids coincide with their vertex means, namely 0.55 and 0.15, respectively. The falsity component is the singleton $(0.1, 0.1, 0.1, 0.1)$, for which

$$V_F = 0.10.$$

Therefore, by Definition 3.7,

$$S_{1,1}(\tilde{N}_{11}) = \frac{1}{3} [0.55 + (1 - 0.15) + (1 - 0.10)] = 0.7667 \approx 0.767.$$

Applying the same scoring procedure to every entry gives the following baseline crisp decision matrix:

$$X = \begin{pmatrix} 0.767 & 0.702 & 0.417 & 0.817 \\ 0.550 & 0.600 & 0.776 & 0.700 \\ 0.450 & 0.667 & 0.659 & 0.833 \\ 0.724 & 0.567 & 0.567 & 0.543 \\ 0.567 & 0.543 & 0.543 & 0.541 \end{pmatrix}. \quad (12)$$

Table 2: Trapezoidal neutrosophic decision matrix D .

Alt.	Upper level criteria		Lower level criteria	
	C1: Contribution to org. performance	C2: Transformation effort	C3: Investment cost	C4: Developer reliability
A1	(0.4,0.5,0.6,0.7)(0.0,0.1,0.2,0.3) (0.1,0.1,0.1,0.1)	(0.3,0.4,0.5,0.5)(0.1,0.2,0.3,0.4) (0.0,0.1,0.1,0.1)	(0.1,0.1,0.1,0.1)(0.1,0.1,0.1,0.1) (0.6,0.7,0.8,0.9)	(0.7,0.7,0.7,0.7)(0.0,0.1,0.2,0.3) (0.1,0.1,0.1,0.1)
A2	(0.0,0.1,0.2,0.3)(0.0,0.1,0.2,0.3) (0.2,0.3,0.4,0.5)	(0.0,0.1,0.1,0.2)(0.0,0.1,0.2,0.3) (0.0,0.1,0.2,0.3)	(0.4,0.5,0.6,0.7)(0.1,0.1,0.1,0.1) (0.0,0.1,0.2,0.2)	(0.4,0.4,0.4,0.4)(0.0,0.1,0.2,0.3) (0.0,0.1,0.2,0.3)
A3	(0.0,0.1,0.1,0.2)(0.1,0.1,0.1,0.1) (0.5,0.6,0.7,0.8)	(0.2,0.3,0.4,0.5)(0.0,0.1,0.2,0.3) (0.1,0.2,0.2,0.3)	(0.2,0.3,0.4,0.5)(0.0,0.1,0.2,0.3) (0.1,0.2,0.3,0.3)	(0.6,0.7,0.7,0.8)(0.1,0.1,0.1,0.1) (0.0,0.1,0.1,0.2)
A4	(0.3,0.4,0.5,0.5)(0.0,0.1,0.2,0.3) (0.0,0.1,0.1,0.2)	(0.1,0.2,0.3,0.4)(0.1,0.1,0.1,0.1) (0.3,0.4,0.5,0.6)	(0.1,0.2,0.3,0.4)(0.1,0.1,0.1,0.1) (0.3,0.4,0.5,0.6)	(0.1,0.2,0.3,0.4)(0.1,0.1,0.1,0.1) (0.4,0.5,0.6,0.6)
A5	(0.1,0.2,0.3,0.4)(0.1,0.1,0.1,0.1) (0.3,0.4,0.5,0.6)	(0.1,0.2,0.3,0.4)(0.1,0.1,0.1,0.1) (0.4,0.5,0.6,0.6)	(0.1,0.2,0.3,0.4)(0.1,0.1,0.1,0.1) (0.4,0.5,0.6,0.6)	(0.1,0.2,0.3,0.3)(0.1,0.2,0.3,0.4) (0.2,0.3,0.4,0.5)

Attribute weight vector: $W = (0.25, 0.25, 0.30, 0.20)^T$

The entries in Eq. 12 are displayed to three decimal places for readability, while the subsequent SYAI calculations are performed using the corresponding unrounded score values. By Theorem 3.15, the (1, 1) representative values coincide with Ye's vertex means for symmetric components. For asymmetric components, the centroid-based values may differ slightly from the corresponding vertex means.

4.2. Ranking Using the SYAI Method

The SYAI normalisation transforms each crisp score into a measure of similarity to the criterion-specific ideal value:

$$N_{ij} = C + (1 - C) \left(1 - \frac{|x_{ij} - x_j^*|}{R_j} \right), \quad C = 0.01, \quad R_j = \max_i x_{ij} - \min_i x_{ij}. \quad (13)$$

For a benefit criterion,

$$x_j^* = \max_i x_{ij},$$

whereas for a cost criterion,

$$x_j^* = \min_i x_{ij}.$$

The normalised values are subsequently weighted according to

$$v_{ij} = w_j N_{ij}. \quad (14)$$

Using the weight vector

$$W = (0.25, 0.25, 0.30, 0.20)^T,$$

the ideal and anti-ideal points are

$$A^+ = (0.25, 0.25, 0.30, 0.20),$$

and

$$A^- = C W = (0.0025, 0.0025, 0.0030, 0.0020).$$

The distances of alternative A_i from the ideal and anti-ideal points are calculated as

$$D_i^+ = \sum_{j=1}^4 |v_{ij} - A_j^+|, \quad D_i^- = \sum_{j=1}^4 |v_{ij} - A_j^-|. \quad (15)$$

The corresponding SYAI closeness index is then defined as

$$D_i = \frac{D_i^-}{D_i^- + D_i^+}. \quad (16)$$

A larger value of D_i indicates an alternative that is closer to the ideal point and farther from the anti-ideal point. The resulting values are reported in Table 3.

Table 3: SYAI closeness scores for the baseline crisp decision matrix at $(\alpha, \beta) = (1, 1)$

Alternative	D_i^+	D_i^-	D_i
A_1	0.2588	0.7312	0.7386
A_2	0.6458	0.3442	0.3477
A_3	0.6409	0.3491	0.3527
A_4	0.3915	0.5985	0.6046
A_5	0.4584	0.5316	0.5370

Based on the closeness indices in Table 3, the baseline ranking is

$$A_1 \succ A_4 \succ A_5 \succ A_3 \succ A_2.$$

For this decision problem, the ranking agrees with that obtained using the classical vertex-mean formulation. This numerical agreement provides an instance-specific confirmation of the backward-compatibility result discussed in Theorem 3.15.

4.3. Sensitivity and Stability Analysis

We evaluate six configurations spanning truth-emphasising to falsity-penalising attitudes. For each, the crisp matrix is regenerated with the parametric score and the SYAI procedure is rerun. Beyond the ordinal ranking, we quantify stability relative to the baseline Φ_1 by Spearman's rank correlation ρ and the total rank displacement $\Delta = \sum_i |r_i - r_i^{(1)}|$, where r_i is the rank of A_i . Table 4 collects the closeness scores and both measures.

Table 4: Closeness scores D_i and ranking stability under six (α, β) configurations

Setting	(α, β)	A_1	A_2	A_3	A_4	A_5	ρ	Δ
Φ_1	(1, 1)	0.7386	0.3477	0.3527	0.6046	0.5370	1.000	0
Φ_2	(1.5, 0.75)	0.7367	0.3467	0.3561	0.6048	0.5346	1.000	0
Φ_3	(2, 0.5)	0.7351	0.3467	0.3590	0.6058	0.5322	1.000	0
Φ_4	(0.75, 1.5)	0.7407	0.3497	0.3493	0.6065	0.5430	0.900	2
Φ_5	(0.5, 2)	0.7426	0.3524	0.3462	0.6097	0.5490	0.900	2
Φ_6	(0.5, 3)	0.7437	0.3538	0.3444	0.6110	0.5540	0.900	2

Note: ρ denotes Spearman's rank correlation coefficient relative to the baseline ranking under Φ_1 , while Δ denotes the total absolute rank displacement from the baseline.

Three points follow. First, the top three positions $A_1 \succ A_4 \succ A_5$ are invariant across every configuration, and A_1 remains the selected alternative throughout, so the decision is robust to the decision-maker's attitude. Second, the parametrization acts precisely where two alternatives are close: A_2 and A_3 differ by only 0.005 at the baseline, and as the falsity sensitivity β grows the gap $D_{A_3} - D_{A_2}$ moves from +0.005 through zero to -0.009 , reversing their order from Φ_4 onward. Third, the stability measures make this precise without relying on the rank plot alone: $\rho = 1$ for the truth-emphasising settings and $\rho = 0.90$ once the reversal occurs, with displacement $\Delta = 2$ confined to the single A_2/A_3 transposition. The effect is deliberately modest because the alternatives are well separated; the parametrization refines the ranking at genuine near-ties rather than manufacturing volatility in a well-conditioned problem.

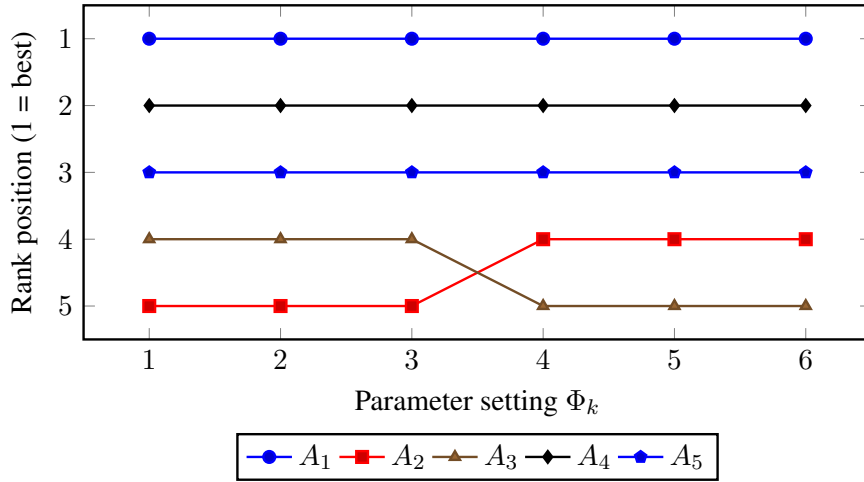


Figure 4: Rank-position sensitivity under the six PTNS shape settings. The top three positions are invariant; only the near-tied pair A_2/A_3 transposes as the falsity sensitivity β increases.

4.4. Second Example and Cross-Method Comparison

To examine the proposed framework on a larger and independent decision problem, and to compare its ranking with those produced by established MADM methods, we consider the selection of a cloud service provider. Six providers,

$$P_1, \dots, P_6,$$

are evaluated with respect to five criteria: cost (C_1 , cost criterion), security (C_2), performance (C_3), scalability (C_4), and support quality (C_5), where C_2 – C_5 are benefit criteria. The corresponding trapezoidal neutrosophic assessments are reported in Table 5.

Applying the centroid-based score in Definition 3.7 at $(\alpha, \beta) = (1, 1)$ gives the crisp decision matrix

$$X^{(2)} = \begin{pmatrix} 0.467 & 0.833 & 0.767 & 0.789 & 0.715 \\ 0.667 & 0.683 & 0.715 & 0.676 & 0.807 \\ 0.607 & 0.724 & 0.683 & 0.769 & 0.676 \\ 0.717 & 0.617 & 0.800 & 0.567 & 0.641 \\ 0.667 & 0.633 & 0.722 & 0.667 & 0.609 \\ 0.783 & 0.533 & 0.567 & 0.750 & 0.691 \end{pmatrix}. \quad (17)$$

The entries in Eq. 17 are displayed to three decimal places. The subsequent calculations should be performed using the corresponding unrounded centroid-based scores.

For comparison, the same crisp decision matrix, attribute weights, and benefit/cost classifications are used with four MADM procedures: PTNS–SYAI, TOPSIS, VIKOR, and COPRAS. This common input structure allows the resulting rankings to be compared without introducing differences arising from alternative defuzzification procedures or weighting schemes. The resulting rankings are presented in Table 6.

As shown in Table 6, all four methods produce the same ordering:

$$P_1 \succ P_3 \succ P_2 \succ P_5 \succ P_4 \succ P_6.$$

Accordingly, the Spearman rank correlation coefficient between PTNS–SYAI and each of TOPSIS, VIKOR, and COPRAS is 1.000.

Table 5: Trapezoidal neutrosophic decision matrix $D^{(2)}$ for cloud service provider selection

Provider	C_1 Cost	C_2 Security	C_3 Performance	C_4 Scalability	C_5 Support
P_1	(0.1, 0.2, 0.2, 0.3)	(0.6, 0.7, 0.8, 0.9)	(0.5, 0.6, 0.7, 0.8)	(0.6, 0.7, 0.7, 0.8)	(0.5, 0.6, 0.7, 0.7)
	(0.1, 0.1, 0.2, 0.2)	(0.0, 0.1, 0.1, 0.2)	(0.1, 0.1, 0.2, 0.2)	(0.1, 0.1, 0.1, 0.2)	(0.1, 0.2, 0.2, 0.3)
	(0.5, 0.6, 0.7, 0.8)	(0.1, 0.1, 0.2, 0.2)	(0.1, 0.2, 0.2, 0.3)	(0.1, 0.2, 0.2, 0.3)	(0.2, 0.2, 0.3, 0.4)
P_2	(0.3, 0.4, 0.5, 0.6)	(0.4, 0.5, 0.6, 0.7)	(0.5, 0.6, 0.6, 0.7)	(0.4, 0.5, 0.6, 0.7)	(0.6, 0.7, 0.8, 0.8)
	(0.1, 0.1, 0.2, 0.2)	(0.1, 0.2, 0.2, 0.3)	(0.1, 0.1, 0.2, 0.3)	(0.1, 0.2, 0.2, 0.3)	(0.0, 0.1, 0.1, 0.2)
	(0.2, 0.3, 0.3, 0.4)	(0.2, 0.3, 0.3, 0.4)	(0.2, 0.2, 0.3, 0.4)	(0.2, 0.3, 0.4, 0.4)	(0.1, 0.2, 0.2, 0.3)
P_3	(0.3, 0.4, 0.4, 0.5)	(0.5, 0.6, 0.6, 0.7)	(0.4, 0.5, 0.6, 0.7)	(0.5, 0.6, 0.7, 0.8)	(0.5, 0.6, 0.6, 0.7)
	(0.1, 0.2, 0.2, 0.3)	(0.1, 0.1, 0.2, 0.2)	(0.1, 0.2, 0.2, 0.3)	(0.0, 0.1, 0.2, 0.2)	(0.1, 0.2, 0.3, 0.3)
	(0.3, 0.3, 0.4, 0.5)	(0.2, 0.2, 0.3, 0.4)	(0.2, 0.3, 0.3, 0.4)	(0.1, 0.2, 0.3, 0.3)	(0.2, 0.3, 0.4, 0.5)
P_4	(0.4, 0.5, 0.6, 0.7)	(0.3, 0.4, 0.5, 0.6)	(0.6, 0.7, 0.7, 0.8)	(0.3, 0.4, 0.4, 0.5)	(0.4, 0.5, 0.5, 0.6)
	(0.1, 0.1, 0.2, 0.2)	(0.1, 0.2, 0.2, 0.3)	(0.0, 0.1, 0.1, 0.2)	(0.2, 0.2, 0.3, 0.3)	(0.1, 0.2, 0.2, 0.3)
	(0.2, 0.2, 0.3, 0.3)	(0.3, 0.4, 0.4, 0.5)	(0.1, 0.2, 0.2, 0.3)	(0.3, 0.4, 0.5, 0.6)	(0.3, 0.3, 0.4, 0.5)
P_5	(0.4, 0.5, 0.5, 0.6)	(0.4, 0.4, 0.5, 0.6)	(0.5, 0.6, 0.7, 0.7)	(0.4, 0.5, 0.6, 0.6)	(0.3, 0.4, 0.5, 0.6)
	(0.1, 0.2, 0.2, 0.3)	(0.1, 0.2, 0.2, 0.3)	(0.1, 0.1, 0.2, 0.3)	(0.1, 0.2, 0.2, 0.3)	(0.1, 0.2, 0.3, 0.3)
	(0.2, 0.3, 0.3, 0.4)	(0.3, 0.3, 0.4, 0.5)	(0.2, 0.2, 0.3, 0.4)	(0.2, 0.3, 0.4, 0.4)	(0.3, 0.4, 0.4, 0.5)
P_6	(0.5, 0.6, 0.7, 0.8)	(0.2, 0.3, 0.4, 0.5)	(0.3, 0.4, 0.4, 0.5)	(0.5, 0.6, 0.7, 0.7)	(0.4, 0.5, 0.6, 0.7)
	(0.0, 0.1, 0.1, 0.2)	(0.2, 0.2, 0.3, 0.3)	(0.2, 0.2, 0.3, 0.3)	(0.1, 0.1, 0.2, 0.2)	(0.1, 0.1, 0.2, 0.3)
	(0.1, 0.2, 0.2, 0.3)	(0.4, 0.5, 0.5, 0.6)	(0.3, 0.4, 0.5, 0.6)	(0.1, 0.2, 0.3, 0.3)	(0.2, 0.3, 0.3, 0.4)

Note: Each cell lists the truth, indeterminacy, and falsity quadruples, respectively. The attribute weights are $W = (0.30, 0.20, 0.20, 0.15, 0.15)$, where C_1 is a cost criterion and C_2 – C_5 are benefit criteria.

Table 6: Cross-method ranking comparison for the second example

Method	P_1	P_2	P_3	P_4	P_5	P_6
PTNS–SYAI	1	3	2	5	4	6
TOPSIS	1	3	2	5	4	6
VIKOR	1	3	2	5	4	6
COPRAS	1	3	2	5	4	6

Note: Spearman’s rank correlation coefficient relative to PTNS–SYAI is 1.000 for TOPSIS, VIKOR, and COPRAS. All four methods use the same centroid-defuzzified matrix $X^{(2)}$ at $(\alpha, \beta) = (1, 1)$.

The complete agreement observed in this example demonstrates cross-method consistency between PTNS–SYAI and the three comparison methods when they are applied to the same centroid-defuzzified decision matrix. The methods nevertheless retain different ranking mechanisms: TOPSIS is based on relative distances from ideal and anti-ideal solutions, VIKOR uses a compromise-ranking structure, whereas COPRAS employs proportional evaluation of benefit and cost criteria. The agreement therefore indicates that, for this particular decision matrix, the SYAI aggregation stage produces the same ordering as these established approaches.

Applying PTNS–SYAI across the six (α, β) configurations considered in the sensitivity analysis also preserves the ordering of the six providers in this example. Thus, within the investigated parameter range, the second example exhibits greater ranking stability than the first example. This result should be interpreted as instance-specific rather than as evidence that parameter changes can affect rankings only when alternatives are nearly tied. Evaluation on larger real-world datasets, including applications in areas such as finance or supply-chain management, remains an appropriate direction for further empirical investigation.

5. Conclusion

This paper introduced a parametrized trapezoidal neutrosophic set (PTNS), in which the ascending and descending segments of each trapezoidal membership function are shaped by two parameters (α, β) . Its central methodological component is a score and accuracy pair derived from the centroid of the parametrized membership function. Because the parameters enter through the membership geometry rather than as external exponents, the three neutrosophic components are defuzzified by a single common operator, and the score rests on a derived rather than heuristic footing. We established that the score is bounded and correctly oriented, monotone in the underlying data, smooth and monotone in (α, β) , predictable under extreme parameter values, and exactly reducible to the classical trapezoidal neutrosophic evaluation at $(\alpha, \beta) = (1, 1)$.

The software-selection example showed that, at $(\alpha, \beta) = (1, 1)$, the crisp matrix and the SYAI ranking coincide with the classical result. Away from the linear case the parametrization adjusts the ranking in a controlled way: the leading alternatives remained fixed across all six tested settings, and the only change was the transposition of a single near-tied pair as the falsity sensitivity increased, reflected in a Spearman correlation that fell from 1.0 to 0.90. The effect is intentionally conservative; the parametrization refines the ranking where alternatives are genuinely close rather than introducing volatility into a well-separated problem.

A second, larger provider-selection problem served to compare the framework against established methods. On this problem the PTNS–SYAI ranking agreed exactly with those produced by TOPSIS, VIKOR, and COPRAS, with unit Spearman correlation in every case. This cross-method agreement validates the aggregation stage and supports the use of SYAI for its simplicity and its smooth response to the parametrization, while confirming that the parametric score can be paired with other ranking methods without loss of discrimination.

Several directions remain open. The most substantial concerns the score itself: a role-separated emphasis operator, in which the shape parameters act asymmetrically on the truth

and falsity components while remaining derived rather than heuristic, would sharpen the behavioural interpretation of (α, β) and is a natural next step. Component-specific parametrization, (α_T, β_T) , (α_I, β_I) , (α_F, β_F) , offers a further generalization. Empirically, validation on real-world datasets from finance, supply-chain management, and engineering design would test the framework beyond constructed examples, and embedding the parametric score within weighting or optimization procedures would extend it to larger and more demanding decision problems.

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Centre of Foundation Studies

Universiti Teknologi MARA, Cawangan Selangor, Kampus Dengkil

43800 Dengkil

Selangor, MALAYSIA

E-mail: azdi20@uitm.edu.my

Fakulti Sains Komputer dan Matematik

Universiti Teknologi MARA (UiTM) Cawangan Negeri Sembilan, Kampus Seremban

70300 Seremban

Negeri Sembilan, MALAYSIA

E-mail: zahari@uitm.edu.my, shafiqkamari922@gmail.com*

Manufacturing and Management Section

Universiti Kuala Lumpur Malaysia Italy Institute

56100 Kuala Lumpur, MALAYSIA

E-mail: zaifilla@unikl.edu.my

Department of Mathematics

Faculty of Education for Pure Sciences

University of Anbar

Ramadi, Anbar, IRAQ

E-mail: faisal.ghazi@uoanbar.edu.iq

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*Corresponding author