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# SINGLY DIAGONALLY IMPLICIT MULTISTEP BLOCK METHOD FOR NUMERICAL SOLUTION OF STIFF ORDINARY DIFFERENTIAL EQUATIONS

(*Kaedah Blok Multilangkah Pepenjuru Tersirat Tunggal untuk Penyelesaian Berangka bagi Persamaan Pembezaan Biasa Kaku*)

HAZIZAH MOHD IJAM*, SAUFIANIM JANA AKSAH & YUSUF DAUDA JIKANTORO

*ABSTRACT*

Stiff ordinary differential equations (ODEs) pose significant challenges in numerical computation due to their rapid variations. Conventional numerical methods often suffer from instability and inefficiency when addressing such equations, which compromises accuracy and hinders progress in mathematical modelling. This research aims to develop an improved numerical method, specifically a singly diagonally implicit multistep block method to effectively solve stiff ODEs. The formulation, which incorporates the block backward differentiation formula (BBDF) generates formulae with absolute stability properties and offers versatility through a free parameter within a specific interval. The diagonally implicit structure characterized by a lower triangular matrix, allows sequential solving with reduced computational costs associated with factorization and back substitution during Newton iteration. The methodology involves approximating solutions at two points simultaneously using a fixed-step approach and conducting comprehensive stability analysis, including zero stability, absolute stability and convergence. Compared to existing BDF methods and MATLAB's stiff solver, the proposed method demonstrates reliability in solving both linear and nonlinear systems across various stiffness levels. This new method proves particularly effective for stiff ODE systems, showcasing its potential for broad scientific and engineering applications.

*Keywords*: singly diagonally implicit; multistep block method; stiff ordinary differential equations

*ABSTRAK*

Persamaan pembezaan biasa (PPB) kaku memberikan cabaran besar dalam pengiraan berangka akibat variasinya yang cepat. Kaedah berangka konvensional sering menghadapi ketidakstabilan dan ketidakcekapan dalam menangani persamaan ini, yang menjejaskan ketepatan dan kemajuan dalam pemodelan matematik. Kajian ini bertujuan untuk membangunkan kaedah berangka yang diperbaik, khususnya kaedah blok multilangkah pepenjuru tersirat tunggal untuk menyelesaikan PPB kaku dengan berkesan. Formulasi kaedah ini, yang menggabungkan formula pembezaan kebelakang blok (FPKB), menghasilkan formula dengan sifat kestabilan mutlak dan menawarkan fleksibiliti melalui parameter bebas dalam julat tertentu. Struktur pepenjuru tersirat yang dicirikan oleh matriks segitiga bawah, membolehkan penyelesaian secara berurutan dengan kos pengiraan yang lebih rendah berkaitan dengan pemfaktoran dan penggantian semula semasa iterasi Newton. Metodologi ini melibatkan penghampiran penyelesaian pada dua titik secara serentak menggunakan pendekatan langkah tetap serta menjalankan analisis kestabilan yang menyeluruh, termasuk kestabilan sifar, kestabilan mutlak dan penumpuan. Berbanding dengan kaedah FPK yang sedia ada dan penyelesai PPB kaku dalam MATLAB, kaedah yang dicadangkan menunjukkan kebolehpercayaan untuk menyelesaikan sistem linear dan tak linear merentasi pelbagai tahap kekakuan. Kaedah baharu ini terbukti amat berkesan untuk sistem PPB kaku, menunjukkan potensinya untuk aplikasi saintifik dan kejuruteraan yang luas.

*Kata kunci*: pepenjuru tersirat tunggal; kaedah blok multilangkah; persamaan pembezaan biasa kaku

## 1. Introduction

Many problems in applied science and engineering are represented using systems of ordinary differential equations (ODEs). These systems often demonstrate stiffness, characterized by solutions that change quickly over different time scales. Traditional numerical techniques can face challenges with instability and a lack of efficiency when applied to such stiff ODEs. This study introduces a singly diagonally implicit multistep block method designed for the efficient numerical solution of stiff ODEs, specifically addressing first-order stiff initial value problems (IVPs) expressed in the form

$$y' = f\left(x, y\right), \quad y\left(a\right) = \mu, \quad x \in \left[a, b\right]. \tag{1}$$

Observe that the function $f\left(x, y\right)$ can be expressed as $Ay + \Phi\left(x\right)$, where $A$ is a matrix of constant dimensions and $\Phi\left(x\right)$ is a continuously differentiable $d$-dimensional vector. As stated in Lambert (1973), Eq. (1) is classified as stiff if the eigenvalues of $s \times s$ matrix $A$, $\lambda_i$ meet the following conditions:

(1) $\mathrm{Re}\left(\lambda_i\right) < 0$,

(2) $\max_i \left|\mathrm{Re}\left(\lambda_i\right)\right| \gg \min_i \left|\mathrm{Re}\left(\lambda_i\right)\right|$, where $\dfrac{\max_i \left|\mathrm{Re}\left(\lambda_i\right)\right|}{\min_i \left|\mathrm{Re}\left(\lambda_i\right)\right|}$ is the stiffness ratio.

Efficient and reliable algorithms for solving stiff initial value problems (IVPs) include the Backward Differentiation Formula (BDF) methods (Soomro *et al.* 2023; Abdi *et al.* 2025; Noor *et al.* 2024), second derivative methods (Eghbaljoo *et al.* 2024; Okor & Nwachukwu 2024), and various Runge-Kutta (RK) methods, including implicit, semi-implicit, and singly implicit formulations (Luan & Alhsmy 2024; Chouchoulis & Schütz 2024). Among these, BDF methods are widely regarded as the most thoroughly tested and well-developed automatic integrators for stiff systems.

Earlier versions of BDF methods primarily utilized numerical differentiation to estimate $y_{n+1}$ at $x_{n+1}$ in each integration step, classifying them as non-block methods. Ibrahim *et al.* (2007) introduced the Block Backward Differentiation Formula (BBDF) to enhance the integration process by approximating a block of values $y_{n+1}, y_{n+2}, \ldots, y_{n+k}$ simultaneously at the corresponding discretization points, $x_{n+1}, x_{n+2}, \ldots, x_{n+k}$. This block-based approach reduces the number of integration steps and computational time while maintaining accuracy, as demonstrated in subsequent studies (Ibrahim *et al.* 2008; Suleiman *et al.* 2014; Rasedee *et al.* 2018).

Subsequent research has further enhanced BBDF's efficiency and accuracy in addressing stiff problems (Nasir *et al.* 2012; Suleiman *et al.* 2013; Ibrahim & Nasarudin 2020; Zawawi & Ibrahim 2020; Soomro *et al.* 2023). The diagonally implicit structure of BBDF reduces computational load for differentiation coefficients, resulting in lower cumulative errors compared to fully implicit methods (Babangida & Musa 2016; Ijam & Ibrahim 2019; Ijam *et* *al.* 2020; Zainuddin *et al.* 2023; Ibrahim *et al.* 2024). Diagonally implicit methods are characterized by a coefficient matrix that is lower triangular, with all entries above the main diagonal being zero. This structure simplifies the solution process compared to fully implicit methods (Ijam & Ibrahim 2019). This approach has evolved into a singly diagonally implicit method, formulated with a lower triangular matrix featuring identical diagonal constants, which reduces computational costs, as demonstrated by Aksah *et al.* (2019). Building on this foundation, this paper aims to advance the theory by introducing the $\rho$-Singly Diagonally Implicit Block Backward Differentiation Formula ($\rho$-SDIBBDF), offering a more refined and comparable approach to previous methods.

## 2. Derivation of the Method

In this section, the $\rho$-SDIBBDF is derived for solving first order stiff IVPs of Eq. (1). Consider a 2-point superclass of the BBDF (2SBBDF) in a fully implicit structure, as proposed by Suleiman *et al.* (2014) presented in the form

$$\sum_{j=0}^{3} \alpha_{j,i}\, y_{n+j-1} = h\beta_{k,i}\left(f_{n+k} - \rho f_{n+k-1}\right), \tag{2}$$

where $k = i = 1, 2$ represent the formula for the first point, $y_{n+1}$ and the second points, $y_{n+2}$, respectively. The constants $\alpha_{j,i}$ and $\beta_{k,i}$ are subject to the conditions, $\alpha_{k,k} = 1$ and $\beta_{k-1,i} = -\rho\beta_{k,i}$. The formula in Eq. (2) incorporates a free parameter, $\rho$ which is bounded within the interval $(-1,1)$ to ensure that the formula adheres to a necessary condition for stiff stability. Babangida and Musa (2016) extended the work in Suleiman *et al.* (2014) by developing a diagonally implicit version of 2SBBDF, where the $y_{n+2}$ term is excluded in the formulation of the first point. Building on the formulations developed in Suleiman *et al.* (2014) and Babangida and Musa (2016), Ijam and Ibrahim (2019) proposed the $\rho$-Diagonally Implicit Block Backward Differentiation Formula ($\rho$-DIBBDF). The $\rho$-DIBBDF differs from the method in Babangida and Musa (2016) by maintaining the same order for both points.

The $\rho$-SDIBBDF method proposed in this paper is adapted from the $\rho$-DIBBDF and SDIBBDF methods developed by Ijam and Ibrahim (2019) and Aksah *et al.* (2019), respectively. It computes approximate solutions for $y_{n+1}$ and $y_{n+2}$ concurrently at the discretization points $x_{n+1}$ and $x_{n+2}$, using two previous values at $x_n$ and $x_{n-1}$. A key innovation is the introduction of a new class of BBDF by setting $\beta_{k-1,k} \neq 0$, selecting a free parameter $\rho$ within $(-1,1)$ and employing a singly diagonally implicit approach to enhance the performance of existing methods. The resulting $\rho$−SDIBBDF method is expressed as

$$\sum_{j=0}^{k+1} \alpha_{k,j-1} y_{n+j-1} = h\beta_{k,k+1}\left(f_{n+k} - \rho f_{n+k-1}\right). \tag{3}$$

For producing $y_{n+1}$ and $y_{n+2}$ simultaneously, the constants $\alpha_{k,j-1}$ and $\beta_{k,k+1}$ satisfy $\alpha_{k,k} = \gamma$, $\beta_{k,k-1} = -\rho\beta_{k,k}$, with $h$ as the fixed step size.

The linear difference operator corresponding to Eq. (3) is given by

$$L_k\left[y(x_n);h\right]=\sum_{j=0}^{k+1}\alpha_{k,j-1}y_{n+j-1}-h\beta_{k,k+1}\left(y'_{n+k}-\rho y'_{n+k-1}\right). \tag{4}$$

Expanding Eq. (4) via Taylor series and subsequently gathering terms related to the derivative of $y$ leads to

$$L_k\left[y(x_n);h\right]=C_0y(x)+C_1hy'(x)+\cdots+C_qh^qy^{(q)}(x). \tag{5}$$

The formulation of Eq. (5) requires the definition of constants $C_q$, which are specified below.

$$\begin{aligned}
C_0 &= \sum_{j=0}^{k+1}\alpha_{k,j-1},\\
C_1 &= \sum_{j=0}^{k+1}\left[\frac{(j-1)}{1!}\alpha_{k,j-1}-\frac{k^{(0)}}{0!}\beta_{k,k+1}+\rho\beta_{k,k+1}\right],\\
&\vdots\\
C_q &= \sum_{j=0}^{k+1}\left[\frac{(j-1)^q}{q!}\alpha_{k,j-1}-\frac{k^{(q-1)}}{(q-1)!}\beta_{k,k+1}+\frac{(k-1)^{(q-1)}}{(q-1)!}\rho\beta_{k,k+1}\right]. \quad q=2,3,\ldots
\end{aligned} \tag{6}$$

To ensure a singly diagonally implicit form, we impose the condition that $\alpha_{k,k}=\gamma,$ where $\alpha_{k,k}$ represents the diagonal entries of the method. Eq. (5) is then solved for $\beta_{k,k+1}=1$ to obtain the coefficients of $\alpha_{k,j-1}$ as listed in Table 1.

Table 1: Coefficients of $\rho$-SDIBBDF

| $k$ | $\alpha_{k,-1}$ | $\alpha_{k,0}$ | $\alpha_{k,1}$ | $\alpha_{k,2}$ |
|---|---|---|---|---|
| 1 | $\frac{\rho+1}{2}$ | $-2$ | $\gamma=-\frac{\rho-3}{2}$ | 0 |
| 2 | 0 | $\frac{\rho+1}{2}$ | $-2$ | $\gamma=-\frac{\rho-3}{2}$ |

Due to its accurate numerical results and optimal stability properties, the parameter $\rho=-\frac{3}{4}$ is selected (Ijam & Ibrahim 2019) and substituted into Eq. (3). The $\rho$-SDIBBDF formula is then expressed in matrix form, highlighting its singly implicit features with $\gamma=\frac{15}{8},$ as follows.

$$\begin{bmatrix}\frac{1}{8} & -2\\ 0 & \frac{1}{8}\end{bmatrix}\begin{bmatrix}y_{n-1}\\ y_n\end{bmatrix}+\begin{bmatrix}\frac{15}{8} & 0\\ -2 & \frac{15}{8}\end{bmatrix}\begin{bmatrix}y_{n+1}\\ y_{n+2}\end{bmatrix}=h\begin{bmatrix}0 & \frac{3}{4}\\ 0 & 0\end{bmatrix}\begin{bmatrix}f_{n-1}\\ f_n\end{bmatrix}+h\begin{bmatrix}1 & 0\\ \frac{3}{4} & 1\end{bmatrix}\begin{bmatrix}f_{n+1}\\ f_{n+2}\end{bmatrix} \tag{7}$$

Eq. (7) is rearranged and the corrector formula for $\rho$-SDIBBDF is presented as follows.

$$
\begin{aligned}
y_{n+1} &= -\frac{1}{15}y_{n-1} + \frac{16}{15}y_n + \frac{2}{5}hf_n + \frac{8}{15}hf_{n+1}, \\
y_{n+2} &= -\frac{1}{15}y_n + \frac{16}{15}y_{n+1} + \frac{2}{5}hf_{n+1} + \frac{8}{15}hf_{n+2}.
\end{aligned} \tag{8}
$$

## 3. Order and Stability of the Method

To evaluate the suitability of a numerical method for solving stiff problems, it is essential to examine its stability. This discussion commences with definitions of key properties: order, consistency, zero stability and *A*-stability. These concepts are fundamental to the analysis and are outlined below.

**Definition 3.1.** The first order linear multistep method (LMM) in Eq. (3) is of order $p$ if $C_0 = C_1 = \cdots = C_p = 0, C_{p+1} \neq 0,$ where $C_{p+1}$ being the error constant (Henrici 1962).

For a LMM to be deemed valid, a key criterion is the convergence of its solution to the true solution as the step size diminishes towards zero. According to Hall and Watt (1976), the LMM in Eq. (3) is convergent if and only if it satisfies the properties of consistency and zero stability. The following consistency conditions given by Lambert (1973) must be fulfilled for this theorem to be satisfied.

**Definition 3.2.** A LMM is considered consistent if its order, denoted by $p$, is greater than or equal to 1 $(p \geq 1)$. Consistency is achieved if and only if these conditions, specific to the method, are met:

$$
\begin{aligned}
&\sum_{j=0}^{k} \alpha_j = 0, \\
&\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j.
\end{aligned}
$$

The characteristic polynomials associated with the LMM, namely the first and second, are defined as follows.

$$
\begin{aligned}
\rho(\zeta) &= \sum_{j=0}^{k} \alpha_j \, \zeta^j, \\
\sigma(\zeta) &= \sum_{j=0}^{k} \beta_j \, \zeta^j.
\end{aligned}
$$

where $\zeta \in \mathbb{C}$ is a dummy variable. The LMM is consistent if and only if $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$. The stability polynomial, which governs the method's stability characteristics, is expressed as

$$
\pi\left(r, \hat{H}\right) = \rho(r) - \hat{H}\,\sigma(r) = 0, \tag{9}
$$

where $\hat{H} = h\lambda$ and $\lambda = \frac{\partial f}{\partial y}$ is a complex parameter.

**Definition 3.3.** An LMM possesses zero stability if its characteristic polynomial's roots satisfy the condition that no root has a modulus greater than unity, and any root with a modulus equal to one is a simple root.

The following definitions, introduced by Dahlquist (1963), are relevant to absolute stability ($A$-stability), as outlined in Definitions 3.4 and 3.5.

**Definition 3.4.** A region $\Re_A$ is considered the region of absolute stability for a given LMM if, for all roots $r_s$ of the stability polynomial, the condition $|r_s| < 1$ holds, where *s* ranges from 1 to *k*.

**Definition 3.5.** An LMM is considered absolutely stable if all roots of its stability polynomial lie within the unit circle when the method is applied to the scalar test equation $y' = \lambda y$, where $\lambda$ is a complex number with a negative real part. This implies that the stability region encompasses the entire left-hand side of the $\hat{H}$ -plane, as depicted in Figure 1.

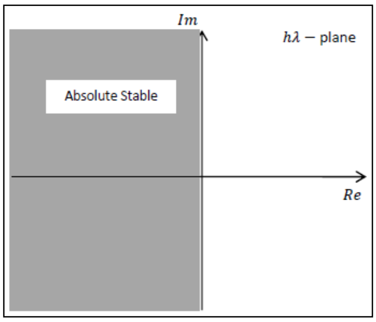



Figure 1: Absolute stability region for an $A$-stable method (Dahlquist 1963)

By substituting the values acquired for $\alpha_{j-1,k}$ and $\beta_{k,k+1}$ into Eq. (6), gives $C_0 = C_1 = C_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and $C_3 = \begin{bmatrix} -\frac{1}{9} & -\frac{1}{9} \end{bmatrix}^T$. It is observed that $C_3 \neq 0$; therefore, the derived method has order 2. In accordance with Definition 3.1, since $\rho$-SDIBBDF is of order 2, it is therefore consistent.

To investigate the stability characteristics, we introduce the scalar test equation $y' = \lambda y$, where $\hat{H} = h\lambda$. Substituting this into Eq. (7) allows us to rewrite the system in a matrix representation, resulting in

$$\underbrace{\begin{bmatrix} 1 - \frac{8}{15}\hat{H} & 0 \\ -\frac{16}{15} - \frac{2}{5}\hat{H} & 1 - \frac{8}{15}\hat{H} \end{bmatrix}}_{A} \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{15} & \frac{16}{15} + \frac{2}{5}\hat{H} \\ 0 & -\frac{1}{15} \end{bmatrix}}_{B} \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}, \tag{10}$$

which is equivalent to $AY_m = BY_{m-1}$, where $Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix}$ and $Y_{m-1} = \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}$. The stability polynomial of the proposed method is obtained by computing the determinant of $|Ar - B| = 0$ and yields

$$\pi\left(r,\hat{H}\right) = r^2 - \frac{16}{15}r^2\hat{H} - \frac{226}{225}r + \frac{64}{225}r^2\hat{H}^2 - \frac{208}{225}r\hat{H} + \frac{1}{225} - \frac{4}{25}r\hat{H}^2. \tag{11}$$

According to Lambert (1973), from Eq. (9), when $\hat{H} = 0$, the roots $r_s$ coincide with the zeros $\zeta_s$ of the first characteristic polynomial $\rho(\zeta)$, which by zero stability, all roots lie in or on the unit circle. Consequently, by substituting $\hat{H} = 0$ and solving Eq. (11) for $r$, yields the roots of $r = 0.004444, 1.$. If the roots of the stability polynomial in Eq. (11) satisfy the conditions outlined in Definition 3.3, then the method is considered zero stable. Given that the method has been shown to be both consistent and zero stable, convergence can be inferred, according to established theorems (Ibrahim *et al.* 2024; Ijam *et al.* 2020).

To determine the stability region's boundary when $\rho = -\frac{3}{4}$, the boundary locus method is employed. This involves setting $r = e^{i\theta}$ in Eq. (11) and solving $\pi\left(r,\hat{H}\right) = 0$ to find the roots $r$. Figure 2 represents the plot of stability region for $\theta \in [0, 2\pi]$ where $|r| < 1$.

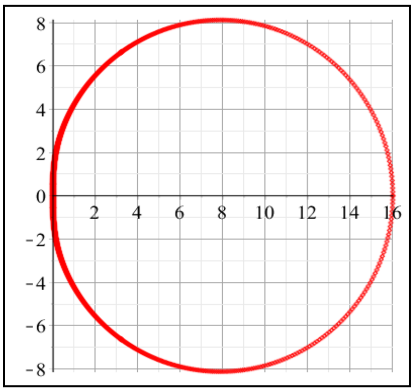



Figure 2: Stability region of $\rho$-SDIBBDF of order 2

As shown in Figure 2, the absolute stability region for the $\rho$-SDIBBDF method is located exterior to the closed contour, with the unstable region contained within. Based on Definitions 3.4 and 3.5, the method exhibits *A*-stability because its region of absolute stability encompasses the entire left half-plane. Consequently, the proposed method is well-suited for solving stiff ODEs.

## 4. Implementation of the Method

To implement the method, Newton's iterative scheme is employed to obtain approximate solutions for $y_{n+1}$ and $y_{n+2}$. Based on Eq. (8), an alternative representation of the equations is

$$F_1 = y_{n+1} - \frac{2}{5}hf_n - \frac{8}{15}hf_{n+1} - \eta_1,$$
$$F_2 = y_{n+2} - \frac{2}{5}hf_{n+1} - \frac{8}{15}hf_{n+2} - \eta_2,$$

where $\eta_1$ and $\eta_2$ are the back values of the formula as follows.

$$\eta_1 = -\frac{1}{15}y_{n-1} + \frac{16}{15}y_n,$$
$$\eta_2 = -\frac{1}{15}y_n + \frac{16}{15}y_{n+1}.$$

In an iterative scheme, let $y_{n+j}^{(i)}$, $j = 1, 2$ be the $i$-th iterate of $y_{n+j}$. We define $e_{n+j}^{(i+1)}$ as the differences between two consecutive approximations,

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}. \tag{13}$$

In its general form, Newton's method updates the approximation as follows.

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - \left[F_j\left(y_{n+j}^{(i)}\right)\right]\left[F_j'\left(y_{n+j}^{(i)}\right)\right]^{-1}, \; j = 1, 2.$$

Hence, from Eqs. (12) and (13), we obtain the following approximation,

$$\left[F_j'\left(y_{n+j}^{(i)}\right)\right]e_{n+j}^{(i+1)} = -\left[F_j\left(y_{n+j}^{(i)}\right)\right], \; j = 1, 2 \tag{14}$$

and the matrix form of Eq. (14) is given by

$$\begin{bmatrix} 1 - \frac{8}{15}h\frac{\partial f_{n+1}^{(i)}}{\partial y_{n+1}^{(i)}} & 0 \\ -\frac{16}{15} - \frac{2}{5}h\frac{\partial f_{n+1}^{(i)}}{\partial y_{n+1}^{(i)}} & 1 - \frac{8}{15}h\frac{\partial f_{n+2}^{(i)}}{\partial y_{n+2}^{(i)}} \end{bmatrix}\begin{bmatrix} e_{n+1}^{(i+1)} \\ e_{n+2}^{(i+1)} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ \frac{16}{15} & -1 \end{bmatrix}\begin{bmatrix} y_{n+1}^{(i)} \\ y_{n+2}^{(i)} \end{bmatrix} + h\begin{bmatrix} 0 & \frac{2}{5} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} f_{n-1}^{(i)} \\ f_n^{(i)} \end{bmatrix}$$
$$+ h\begin{bmatrix} \frac{8}{15} & 0 \\ \frac{2}{5} & \frac{8}{15} \end{bmatrix}\begin{bmatrix} f_{n+1}^{(i)} \\ f_{n+2}^{(i)} \end{bmatrix} + \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}.$$

In accordance with standard LMM terminology, the method is implemented using a *PECE* sequence, where *P* and *C* represent the predictor and corrector stages, respectively and *E* indicates the function evaluation. The algorithm of the proposed method in the code is provided as follows.

Step 1: Predictor (P):

Initial approximations are obtained using predictor formula, $y_{n+j}^{(p)}$.

$$y_{n+1}^{(p)} = -y_{n-1} + 2y_n, \quad y_{n+2}^{(p)} = -2y_{n-1} + 3y_n.$$

Step 2: Evaluation (E): Function *f* is evaluated at $\left(x_{n+j}, y_{n+j}^{(p)}\right)$.

Step 3: Corrector (C): A more accurate solution is computed using corrector formula, $y_{n+j}^{(c)}$.

Step 4: Evaluation (E): Function *f* is evaluated again at $\left(x_{n+j}, y_{n+j}^{(c)}\right)$.

## 5. Numerical Experiments

To assess the effectiveness of the proposed method, we analyze three test problems involving first-order stiff ODEs. The evaluation is based on a comparison of the absolute errors, calculated as follows.

$$E_i = \left| y_i - y\left(x_i\right) \right|$$

In this context, $y_i$ represents the approximate solution, while $y\left(x_i\right)$ signifies the corresponding exact solution. For test problems 2 and 3, where the theoretical solutions are unknown, the approximation values are compared with results obtained using MATLAB solver, ode15s. The performance of the $\rho$-SDIBBDF method (order 2) is evaluated against the Second Derivative Generalized Backward Differentiation Formulae of orders 4 and 5 ($SDGBDF_4$ and $SDGBDF_5$), as described by Nwachukwu and Okor (2018). For consistency, $\rho$-SDIBBDF is evaluated alongside other high-order accuracy versions of the BDF method, which are well-established in addressing stiff problems and provide a relevant basis for performance assessment.

**Test Problem 1:** Consider a singular pertubated problem in Nwachukwu and Okor (2018) with corresponding initial conditions.

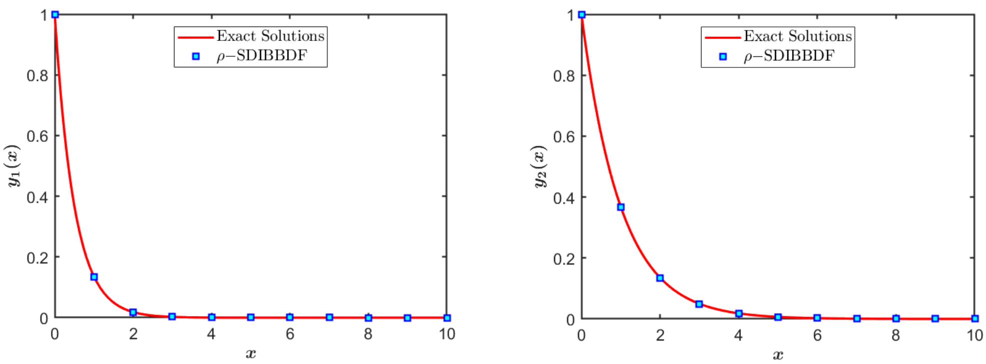
$$y_1' = -\left(2 + 10^4\right) y_1 + 10^4 y_2^2, \qquad y_1(0) = 1$$

$$y_2' = y_1 - y_2\left(1 + y_2\right), \qquad y_2(0) = 1$$

Exact solutions: $y_1(x) = e^{-2x}, \quad y_2(x) = e^{-x}.$

Table 2: Absolute error, $E_i$ for Test Problem 1 at $h = 0.01$

| $x$ | $y_i$ | $E_i$, $i=1,2$ | | |
|---|---|---|---|---|
| | | $\rho$-SDIBBDF | SDGBDF$_4$ | SDGBDF$_5$ |
| 1.0 | $y_1$ | 4.48089e-09 | 3.06126e-11 | 3.43744e-11 |
| | $y_2$ | 6.08973e-09 | 4.22623e-11 | 4.96455e-11 |
| 2.0 | $y_1$ | 6.50055e-10 | 1.03235e-11 | 5.15573e-12 |
| | $y_2$ | 2.40147e-09 | 3.96899e-11 | 1.91052e-11 |
| 3.0 | $y_1$ | 9.38799e-11 | 2.39019e-12 | 6.49362e-13 |
| | $y_2$ | 9.42744e-10 | 2.40044e-11 | 6.79007e-12 |
| 4.0 | $y_1$ | 1.35043e-11 | 4.31932e-13 | 9.57145e-14 |
| | $y_2$ | 3.68627e-10 | 1.20298e-11 | 2.61306e-12 |
| 5.0 | $y_1$ | 1.93551e-12 | 8.00396e-14 | 1.20228e-14 |
| | $y_2$ | 1.43619e-10 | 5.82196e-12 | 9.28587e-13 |
| 6.0 | $y_1$ | 2.76524e-13 | 1.27167e-14 | 1.77133e-15 |
| | $y_2$ | 5.57809e-11 | 2.56518e-12 | 3.57306e-13 |
| 7.0 | $y_1$ | 3.93907e-14 | 2.18299e-15 | 2.22482e-16 |
| | $y_2$ | 2.16045e-11 | 1.15005e-12 | 1.26970e-13 |
| 8.0 | $y_1$ | 5.59495e-15 | 3.27871e-16 | 3.27798e-17 |
| | $y_2$ | 8.34661e-12 | 4.79014e-13 | 4.88578e-14 |
| 9.0 | $y_1$ | 7.92227e-16 | 4.83562e-17 | 4.11682e-18 |
| | $y_2$ | 3.21778e-12 | 1.95919e-13 | 1.73602e-14 |
| 10.0 | $y_1$ | 1.11652e-16 | 7.87909e-18 | 6.06524e-19 |
| | $y_2$ | 1.23792e-12 | 8.33725e-14 | 6.67981e-15 |




a) Graph for $y_1(x)$ b) Graph for $y_2(x)$

Figure 3: Solution curves of $\rho$-SDIBBDF for Test Problem 1

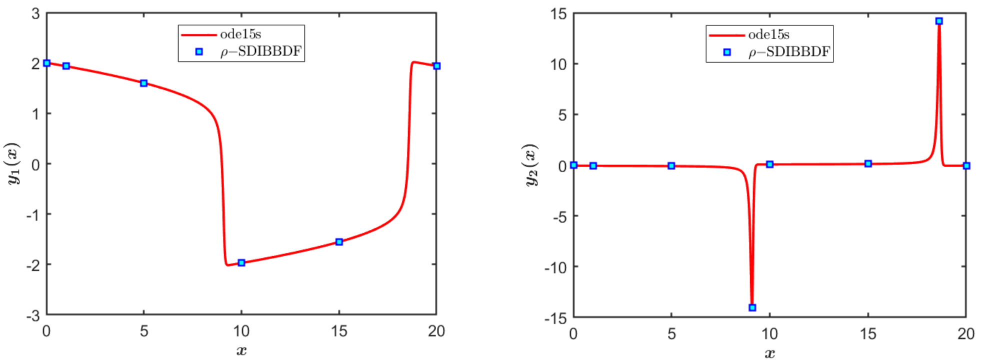
**Test Problem 2:** Consider a nonlinear problem known as Van der Pol equations in Nwachukwu and Okor (2018).

$$y_1' = y_2, \qquad y_1(0) = 2$$

$$y_2' = -y_1 + 10y_2\left(1 + y_1^2\right), \qquad y_2(0) = 0$$

Table 3: Absolute error, $E_i$ for Test Problem 2 at $h = 0.001$

| $x$ | $y_i$ | $E_i,\ i=1,2$ | | |
|---|---|---|---|---|
| | | $\rho$-SDIBBDF | $\text{SDGBDF}_4$ | $\text{SDGBDF}_5$ |
| 1.0 | $y_1$ | 6.97310e-05 | 1.32308e-04 | 1.82747e-05 |
| | $y_2$ | 4.42059e-06 | 8.31878e-06 | 1.09583e-06 |
| 5.0 | $y_1$ | 2.23451e-04 | 3.36208e-04 | 1.47667e-04 |
| | $y_2$ | 6.44750e-06 | 2.22430e-05 | 3.98291e-06 |
| 10.0 | $y_1$ | 1.72177e-04 | 3.67025e-05 | 2.25767e-04 |
| | $y_2$ | 1.01592e-05 | 1.98333e-06 | 1.32565e-05 |
| 15.0 | $y_1$ | 7.77917e-04 | 2.46188e-05 | 6.92917e-04 |
| | $y_2$ | 9.10759e-05 | 3.09782e-05 | 7.75023e-05 |
| 20.0 | $y_1$ | 2.58728e-04 | 9.04906e-05 | 2.00872e-04 |
| | $y_2$ | 1.05413e-05 | 1.10280e-05 | 6.98618e-06 |




a) Graph for $y_1(x)$ b) Graph for $y_2(x)$

Figure 4: Solution curves of $\rho$-SDIBBDF for Test Problem 2

**Test Problem 3:** Consider a nonlinear chemistry problem known as Robertson's equations in Nwachukwu and Okor (2018).

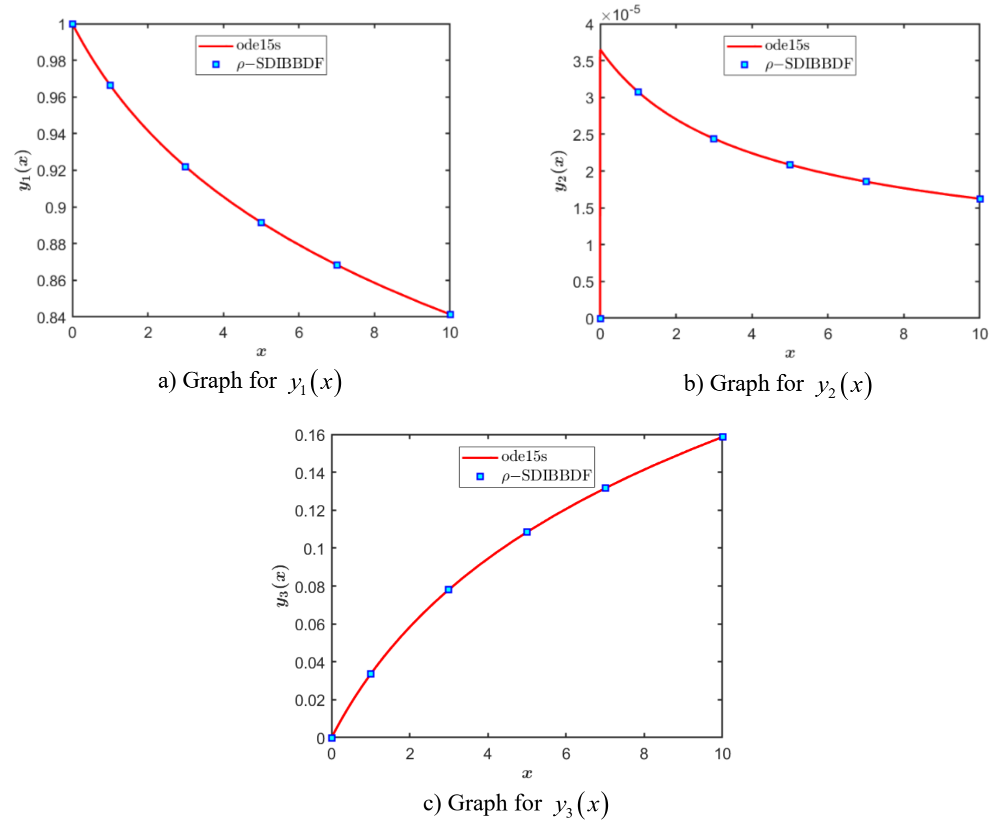
$$
\begin{aligned}
y_1' &= -0.04y_1 + 10^4 y_2 y_3, & y_1(0) &= 1 \\
y_2' &= 0.04y_1 - 10^4 y_2 y_3 - 3\times 10^7 y_2^{\,2}, & y_2(0) &= 0 \\
y_3' &= 3\times 10^7 y_2^{\,2} & y_3(0) &= 0
\end{aligned}
$$

Table 4: Absolute error, $E_i$ for Test Problem 3 at $h = 0.0001$

| $x$ | $y_i$ | $E_i,\ i=1,2,3$ | | |
|---|---|---|---|---|
| | | $\rho$-SDIBBDF | $\text{SDGBDF}_4$ | $\text{SDGBDF}_5$ |
| 1.0 | $y_1$ | 2.41865e-06 | 2.31233e-05 | 6.11248e-06 |
| | $y_2$ | 3.85010e-10 | 3.68698e-09 | 9.74789e-10 |
| | $y_3$ | 2.41904e-06 | 2.31270e-05 | 6.11346e-06 |

Table 4 (Continued)

| | | | | |
|---|---|---|---|---|
| 3.0 | $y_1$ | 3.57673e-06 | 3.91192e-06 | 3.91187e-06 |
| | $y_2$ | 4.57806e-10 | 4.93245e-10 | 4.93239e-10 |
| | $y_3$ | 3.57719e-06 | 3.91241e-06 | 3.91236e-06 |
| 5.0 | $y_1$ | 7.07186e-06 | 1.02179e-06 | 1.02175e-06 |
| | $y_2$ | 7.43595e-10 | 2.95023e-10 | 2.95027e-10 |
| | $y_3$ | 7.07260e-06 | 1.02149e-06 | 1.02146e-06 |
| 7.0 | $y_1$ | 1.62114e-05 | 4.48772e-05 | 4.48771e-05 |
| | $y_2$ | 1.56153e-09 | 4.22263e-09 | 4.22262e-09 |
| | $y_3$ | 1.62130e-05 | 4.48814e-05 | 4.48814e-05 |
| 10.0 | $y_1$ | 2.30138e-05 | 7.35061e-05 | 7.35060e-05 |
| | $y_2$ | 1.85586e-09 | 5.76555e-09 | 5.76555e-09 |
| | $y_3$ | 2.30156e-05 | 7.35118e-05 | 7.35118e-05 |




a) Graph for $y_1(x)$

b) Graph for $y_2(x)$

c) Graph for $y_3(x)$

Figure 5: Solution curves of $\rho$-SDIBBDF for Test Problem 3

All test problems were successfully solved by Nwachukwu and Okor (2018) using the SDGBDF method of orders 4 and 5. The numerical results are presented in Tables 2, 3, and 4, showcasing various values of the independent variable, $x$ and step size, $h$. The study's findings suggest that the proposed method of order 2 is a viable approach for stiff problems, with performance comparable to that of existing second-derivative methods, including the higher-order SDGBDF.

Furthermore, note that the solution curves of the proposed method for Test Problem 1 align perfectly with the exact solutions shown in Figure 3. Similarly, for Test Problems 2 and 3, the solution curves match the approximate solutions provided by the MATLAB solver in Figures 4 and 5.

## 6. Conclusion

In conclusion, this paper introduces a class of BBDF methods, namely $\rho$-SDIBBDF, which exhibits desirable numerical properties, including zero stability, $A$-stability, and convergence. Despite being a lower-order method, the proposed approach performs comparably to reliable benchmark solvers, such as MATLAB's ode15s, and existing high-order accuracy methods within the BDF family. These characteristics, combined with its consistent performance, establish $\rho$-SDIBBDF as a suitable and effective solver for first order stiff ODEs.

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*Faculty of Computer and Mathematical Sciences*
*Universiti Teknologi MARA*
*40450 Shah Alam*
*Selangor, MALAYSIA*
*E-mail: hazizahijam@uitm.edu.my**

*Centre of Foundation Studies*
*Universiti Teknologi MARA, Cawangan Selangor, Kampus Dengkil*
*43800 Dengkil*
*Selangor, MALAYSIA*
*E-mail: saufianim@uitm.edu.my*

*Applied Numerical Analysis (ApNA) Research Initiative Group (RIG)*
*Faculty of Computer and Mathematical Sciences*
*Universiti Teknologi MARA*
*40450 Shah Alam,*
*Selangor, MALAYSIA*
*E-mail: hazizahijam@uitm.edu.my*, saufianim@uitm.edu.my*

*Department of Mathematical Sciences*
*Ibrahim Badamasi Babangida University*
*911101 Lapai, NIGERIA*
*E-mail: ydjikantoro@ibbu.edu.ng*

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*Corresponding author